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NASA TM X-53294

(NASA-TM-X-53294) METHODS OF COMPUTING THE TRANSFORMATION MATRIX ASSOCIATED WITH GIMBALLESS INERTIAL MEASUREMENT UNITS (NASA) 61 p
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METHODS OF COMPUTING THE TRANSFORMATION MATRIX ASSOCIATED WITH GIMBALLESS INERTIAL MEASUREMENT UNITS

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METHODS OF COMPUTING THE TRANSFORMATION MATRIX ASSOCIATED WITH GIMBALLESS INERTIAL MEASUREMENT UNITS

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Dexter H. Burdeshaw

George C. Marshall Space Flight Center
Huntsville, Alabama

ABSTRACT

Four basic approaches to the calculation of the transformation matrix made necessary by a gimballess inertial measurement unit are discussed. Equations that could be programed for an onboard digital computer are derived for each method to accomplish the transformation from a vehicle-fixed reference frame to an inertial reference frame. The methods described are direction cosines, Euler angles, redundant Euler angles, and four parameters. Some of the advantages and disadvantages of each are discussed.

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July 13, 1965

METHODS OF COMPUTING THE TRANSFORMATION MATRIX ASSOCIATED WITH GIMBALLESS INERTIAL MEASUREMENT UNITS

By

Dexter H. Burdeshaw

ASTRIONICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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METHODS OF COMPUTING THE TRANSFORMATION MATRIX ASSOCIATED WITH GIMBALLESS INERTIAL MEASUREMENT UNITS

SUMMARY

Four basic approaches to the calculation of the transformation matrix made necessary by a gimballess inertial measurement unit are discussed. Equations that could be programed for an onboard digital computer are derived for each method to accomplish the transformation from a vehicle-fixed reference frame to an inertial reference frame. The methods described are direction cosines, Euler angles, redundant Euler angles, and four parameters. Some of the advantages and disadvantages of each are discussed.

SECTION I. INTRODUCTION

It seems probable that some future space vehicles will ge guided with position and velocity information derived from a gimballess inertial measurement unit (IMU). The elimination of the gimbaled IMU and the mounting of gyros and accelerometers directly to the vehicle frame will result in a more flexible packaging configuration, less weight, lower power requirements, and probably greater reliability. Other significant advantages are possible for such a system over conventional gimbaled IMU's, however, whether or not the accuracy will be as good is still questionable. For applications where the accuracy of the gimballess IMU is sufficient, it is probably the best choice. See Figure 1 for a pictorial comparison.

The information needed for navigation and guidance of space vehicles must be known relative to a space-fixed coordinate system. Since a gimballess IMU is mounted to the vehicle frame the sensors will measure relative to a rotating body-fixed coordinate system. Basically, this requires a rotational coordinate transformation. Several methods have been proposed for the calculation of the transformation from the vehicle to a space-fixed coordinate system. These can be grouped into four basic methods which will be considered in this report.

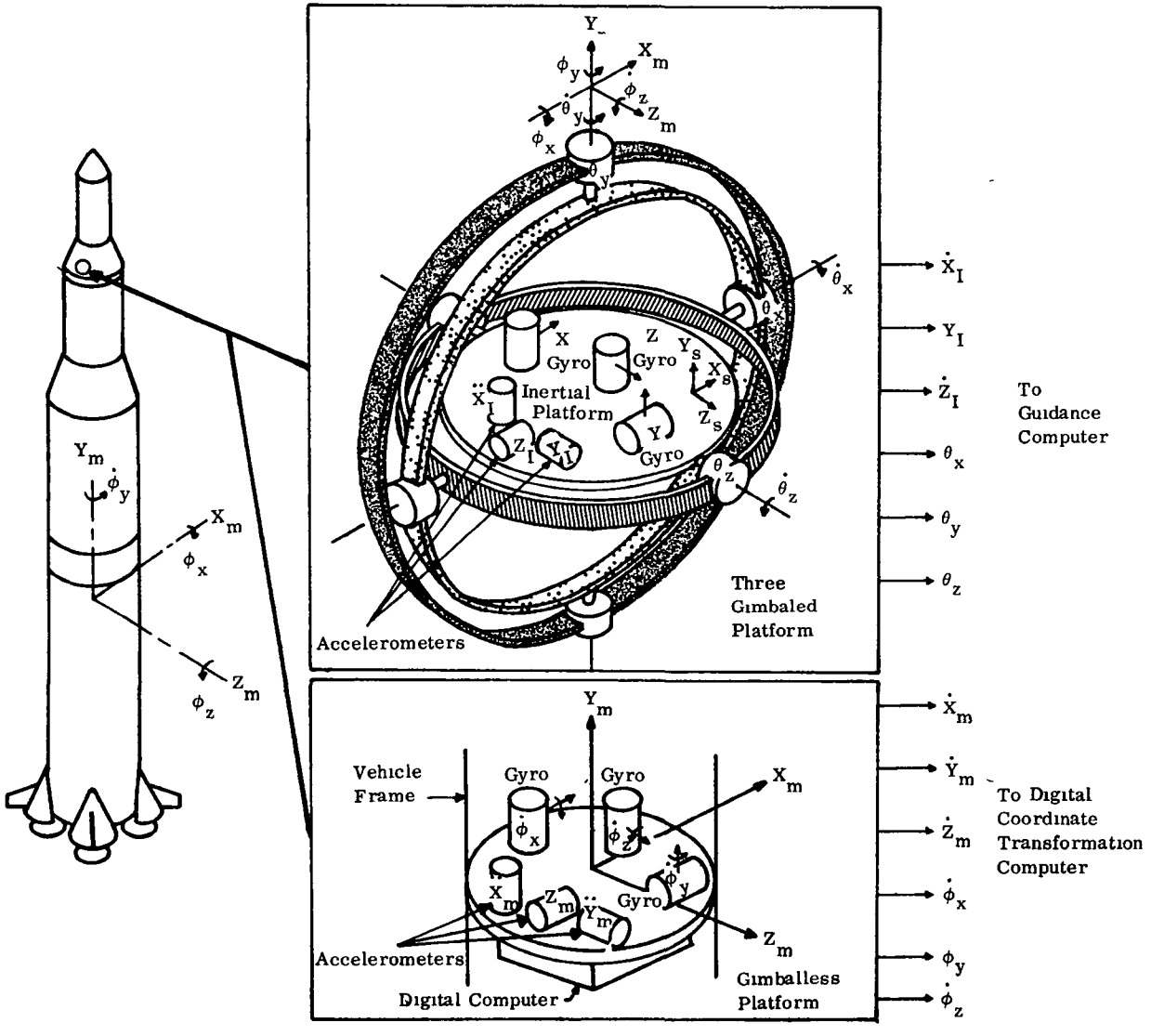


FIGURE 1. TWO TYPES OF PLATFORMS

The possibility of obtaining guidance information in this manner has been known for some time. The computational requirements of this transformation has been impractical for an onboard computer, but with recent advances in digital flight computer state of the art it has become feasible. Another problem has been the accuracy of sensors, especially when subjected to the full range of vehicle motions. Advances in sensor design have helped to reduce this problem also.

SECTION II STATEMENT OF THE PROBLEM

This paper is concerned with the calculation of the transformation that carries the components of velocity in the vehicle coordinate frame into an inertial reference frame for subsequent navigation and guidance calculations. The various methods of calculating the transformation outlined in this report will be studied further using both digital computer and digital differential analyzer techniques to determine the "best" method. Consideration will be given to complexity of the implementation of the method as well as to accuracy.

The basic relationship between measured velocity in the vehicle coordinate system $\vec{V}_m$ and inertial velocity in the space-fixed inertial coordinate system $\vec{V}_I$ is expressed by the equation:

$$\vec{V}_I = C \cdot \vec{V}_m$$

where C is the transformation matrix. Since for guidance the components of inertial velocity must be known in a space-fixed coordinate system, the C matrix must be computed throughout the powered phase of a space flight. The calculation of C depends upon information on body rates obtained from body mounted rate gyros. The rate gyro outputs may be processed in several different ways to obtain the transformation. Of these, four basic methods of arriving at the transformation will be considered here: direction cosines, Euler angles, redundant Euler angles, and four parameters.

SECTION III THE TRANSFORMATION MATRIX

The transformation of the components of velocity measured in the vehicle coordinate frame to the inertial frame may be represented as follows·

$$\begin{pmatrix} \dot{X}_I \\ \dot{Y}_I \\ \dot{Z}_I \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{X}_m \\ \dot{Y}_m \\ \dot{Z}_m \end{pmatrix} ,$$

where C is an orthogonal transformation matrix. The derivation of the C matrix is given in Appendix A. The origins of the two coordinate systems are assumed coincident, with the vehicle coordinate frame rotated in some arbitrary manner with respect to the inertial frame at any instant of time. The inertial or space-fixed frame is in reality fixed only in direction with its origin at the instantaneous center of gravity of the vehicle. The elements of the C matrix are the direction cosines of the angles between the coordinate axes of the two systems. For example, c_{11} is the cosine of the least angle between the positive X_m and X_I axes. Figure 2 illustrates this relationship. The elements, c_{11}, ..., c_{33} must be determined from the rate gyro outputs by some scheme to effect the transformation (Fig. 2)

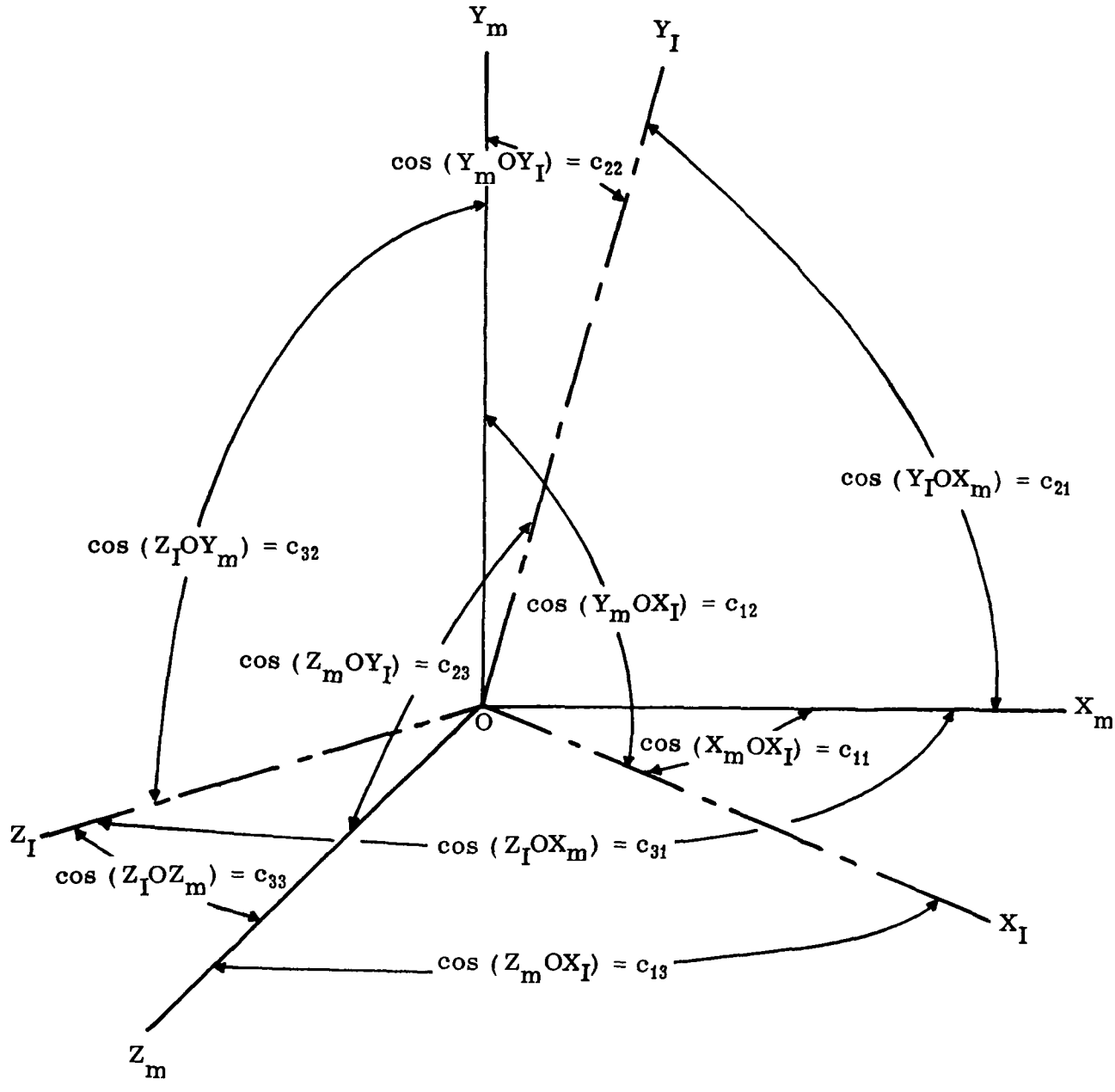
Since C is an orthogonal transformation matrix, there is a set of orthogonality and normality conditions that the elements of C must meet. These are derived in Appendix B. In any implementation to calculate C, the orthogonality and normality conditions must be maintained to some predetermined acceptable degree of accuracy.

SECTION IV DIRECTION COSINE METHOD

The most straightforward way to calculate the elements of C is by determining the rate of change of the direction cosines themselves as the vehicle rotates and by integrating the rates. To illustrate this method define $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ as unit vectors in the inertial coordinate system and $\vec{i}_m$, $\vec{j}_m$, $\vec{k}_m$ as unit vectors in the vehicle coordinate system. The vector representing the rate of vehicle rotation may be expressed:

$$\dot{\vec{\phi}} = \dot{\phi}_x \vec{i}_m + \dot{\phi}_y \vec{j}_m + \dot{\phi}_z \vec{k}_m ,$$

where $\dot{\phi}_x$, $\dot{\phi}_y$, $\dot{\phi}_z$ are the outputs of ideal rate gyros.



$$\begin{pmatrix} \dot{X}_I \\ \dot{Y}_I \\ \dot{Z}_I \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{X}_m \\ \dot{Y}_m \\ \dot{Z}_m \end{pmatrix}$$

FIGURE 2. THE TRANSFORMATION MATRIX AS A MATRIX OF DIRECTION COSINES

The relationship between unit vectors in the two coordinate systems may be expressed·

$$\begin{pmatrix} \vec{i}_I \\ \vec{j}_I \\ \vec{k}_I \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \vec{i}_m \\ \vec{j}_m \\ \vec{k}_m \end{pmatrix}.$$

Differentiate both sides of the equation above with respect to time:

$$\begin{pmatrix} \dot{\vec{i}}_I \\ \dot{\vec{j}}_I \\ \dot{\vec{k}}_I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{c}_{11} & \dot{c}_{12} & \dot{c}_{13} \\ \dot{c}_{21} & \dot{c}_{22} & \dot{c}_{23} \\ \dot{c}_{31} & \dot{c}_{32} & \dot{c}_{33} \end{pmatrix} \begin{pmatrix} \vec{i}_m \\ \vec{j}_m \\ \vec{k}_m \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{\vec{i}}_m \\ \dot{\vec{j}}_m \\ \dot{\vec{k}}_m \end{pmatrix}.$$

Since the unit vectors in the inertial system change neither length nor direction, their time derivative must be zero. The relationship simplifies to the following form:

$$\begin{pmatrix} \dot{c}_{11} & \dot{c}_{12} & \dot{c}_{13} \\ \dot{c}_{21} & \dot{c}_{22} & \dot{c}_{23} \\ \dot{c}_{31} & \dot{c}_{32} & \dot{c}_{33} \end{pmatrix} \begin{pmatrix} \vec{i}_m \\ \vec{j}_m \\ \vec{k}_m \end{pmatrix} = - \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \frac{d}{dt}(\vec{i}_m) \\ \frac{d}{dt}(\vec{j}_m) \\ \frac{d}{dt}(\vec{k}_m) \end{pmatrix}.$$

The $\frac{d}{dt}(\vec{i}_m)$ is the instantaneous rate of change of the unit vector caused by the instantaneous rotation of the vehicle. Therefore, from the definition of the derivative of a vector, since the length of the unit vector is 1 and does not change, the scalar part of the derivative is zero and

$$\frac{d}{dt}(\vec{i}_m) = \dot{\vec{\phi}} \times \vec{i}_m = \begin{vmatrix} \vec{i}_m & \vec{j}_m & \vec{k}_m \\ \dot{\phi}_x & \dot{\phi}_y & \dot{\phi}_z \\ 1 & 0 & 0 \end{vmatrix} = \vec{j}_m \dot{\phi}_z - \vec{k}_m \dot{\phi}_y .$$

Similarly·

$$\frac{d}{dt}(\vec{j}_m) = \dot{\vec{\phi}} \times \vec{j}_m = \begin{vmatrix} \vec{i}_m & \vec{j}_m & \vec{k}_m \\ \dot{\phi}_x & \dot{\phi}_y & \dot{\phi}_z \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}_m \dot{\phi}_x - \vec{i}_m \dot{\phi}_z ,$$

and

$$\frac{d}{dt}(\vec{k}_m) = \dot{\vec{\phi}} \times \vec{k}_m = \begin{vmatrix} \vec{i}_m & \vec{j}_m & \vec{k}_m \\ \dot{\phi}_x & \dot{\phi}_y & \dot{\phi}_z \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}_m \dot{\phi}_y - \vec{j}_m \dot{\phi}_x .$$

Substituting these expressions in the general equation:

$$\begin{pmatrix} \dot{c}_{11} & \dot{c}_{12} & \dot{c}_{13} \\ \dot{c}_{21} & \dot{c}_{22} & \dot{c}_{23} \\ \dot{c}_{31} & \dot{c}_{32} & \dot{c}_{33} \end{pmatrix} \begin{pmatrix} \vec{i}_m \\ \vec{j}_m \\ \vec{k}_m \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \vec{k}_m \dot{\phi}_y - \vec{j}_m \dot{\phi}_z \\ \vec{i}_m \dot{\phi}_z - \vec{k}_m \dot{\phi}_x \\ \vec{j}_m \dot{\phi}_x - \vec{i}_m \dot{\phi}_y \end{pmatrix} .$$

Next the two matrices on each side of this equation are multiplied together and the corresponding elements in the resulting matrix are set equal. Then, by rearranging terms and setting equal the coefficients of the corresponding unit vectors on both sides of the equation, the following set of equations for the nine time derivatives of the elements of the C matrix results in:

$$\dot{c}_{11} = c_{12}\dot{\phi}_z - c_{13}\dot{\phi}_y$$

$$\dot{c}_{12} = c_{13}\dot{\phi}_x - c_{11}\dot{\phi}_z$$

$$\dot{c}_{13} = c_{11}\dot{\phi}_y - c_{12}\dot{\phi}_x$$

$$\dot{c}_{21} = c_{22}\dot{\phi}_z - c_{23}\dot{\phi}_y$$

$$\dot{c}_{22} = c_{23}\dot{\phi}_x - c_{21}\dot{\phi}_z$$

$$\dot{c}_{23} = c_{21}\dot{\phi}_y - c_{22}\dot{\phi}_y$$

$$\dot{c}_{31} = c_{32}\dot{\phi}_z - c_{33}\dot{\phi}_y$$

$$\dot{c}_{32} = c_{33}\dot{\phi}_x - c_{31}\dot{\phi}_z$$

$$\dot{c}_{33} = c_{31}\dot{\phi}_y - c_{32}\dot{\phi}_x \quad .$$

Using these equations in the digital flight computer to calculate the instantaneous rate of change of the elements of C, and then performing an integration to get updated values for the elements themselves, gives one method of computing the transformation. From a practical computation viewpoint, nine integrations are performed to calculate the elements of the C matrix. Theoretically, some of the integrations could be replaced with algebraic computations, but this is seldom proposed.

A further problem is that since C is an orthogonal transformation matrix, the elements of C must satisfy the orthogonality and normality conditions derived in Appendix B within some preassigned tolerance. The continuous calculation of C, by nine integrations, will cause C to drift further and further from satisfying the orthogonality and normality conditions. This problem may be alleviated by first solving for the elements of C by integration as before and then by using the orthogonality and normality conditions to derive a correction factor for each element. An alternative is to solve for some of the elements by integration and for the rest by applying the equations relating the elements derived in Appendix B. Further, a scheme could be set up to alternate the computation of sets of the elements between integration and the algebraic equations.

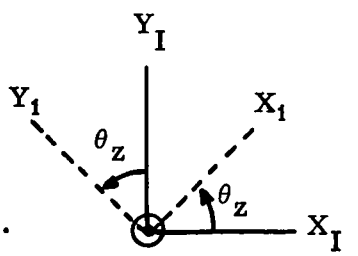
SECTION V EULER ANGLE METHOD

A second method of caloulating the transformation matrix is through three rotations called Euler angles. Euler derived a method of rotating a rigid body successively about its axes so that after the third rotation the original axis system is parallel to a second specified axis system. Therefore, by means of Euler angles, we may relate the orientation of the vehicle body axes to space-fixed axes. Care must be taken in the use of Euler angles since they do not obey the commutative law. This means that the same sequence of rotations must be maintained. However, the particular sequence chosen is arbitrary and may be defined to meet the needs of a particular problem.

We shall define a fixed sequence of rotations to make the transformation from space-fixed to vehicle-fixed axes. The transformation from vehicle-fixed to space-fixed axes is just the inverse of this. The sequence of rotations chosen corresponds to the order of the gimbals on the ST124-M inertial platform to be used with Saturn V. The sequence is first about the inertial Z axis, then about the new X axis, and finally about the resulting Y axis into alignment with the body-fixed coordinate system.

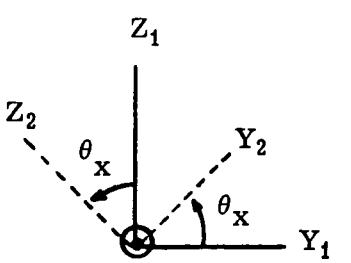
The first rotation is called θ_z and is about the space-fixed Z axis, Z_I:

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_I \\ Y_I \\ Z_I \end{pmatrix}$$



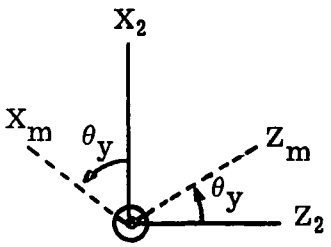
The second rotation is called θ_x and is about the new X_1 axis·

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$



The third rotation is called θ_y and is about the new Y_2 axis:

$$\begin{pmatrix} X_m \\ Y_m \\ Z_m \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix}.$$



The complete transformation from space-fixed to vehicle-fixed coordinate systems can be achieved by multiplying together these three matrices. If $d\theta$ is the product, then:

$$\begin{pmatrix} X_m \\ Y_m \\ Z_m \end{pmatrix} = \begin{pmatrix} d\theta \end{pmatrix} \begin{pmatrix} X_I \\ Y_I \\ Z_I \end{pmatrix}.$$

If C represents the inverse of $d\theta$, then

$$\begin{pmatrix} X_I \\ Y_I \\ Z_I \end{pmatrix} = \begin{pmatrix} C \end{pmatrix} \begin{pmatrix} X_m \\ Y_m \\ Z_m \end{pmatrix}.$$

Since $d\theta$ is orthogonal, to determine C, we may multiply out the three rotation matrices to obtain $d\theta$ and then take the transpose of $d\theta$; or alternately, we may multiply the transpose of each of the rotation matrices in reverse order. Either way the result will be the same. Here, we will determine C by multiplying the transpose of each of the three rotation matrices in reverse order.

$$C = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix},$$

which gives when multiplied out·

$$C = \begin{pmatrix} \cos\theta_z \cos\theta_y & -\sin\theta_z \cos\theta_x & \cos\theta_z \sin\theta_y \\ -\sin\theta_z \sin\theta_x \sin\theta_y & & + \sin\theta_z \sin\theta_x \cos\theta_y \\ \sin\theta_z \cos\theta_y & \cos\theta_z \cos\theta_x & \sin\theta_z \sin\theta_y \\ + \cos\theta_z \sin\theta_x \sin\theta_y & & - \cos\theta_z \sin\theta_x \cos\theta_y \\ -\cos\theta_x \sin\theta_y & \sin\theta_x & \cos\theta_x \cos\theta_y \end{pmatrix}.$$

During flight the vehicle will experience angular rotation rates about its body axes. These angular rotation rates about the body axes may also be expressed as angular rotation rates about the calculated Euler angle axes. All that is necessary is the transformation from the vehicle-fixed reference frame to the reference frame formed by the three Euler angle axes. Since the three Euler angle axes are not, in general, orthogonal, the transformation matrix will not, in general, be orthogonal.

To derive the needed transformation, it is most convenient to work in reverse order. That is, it is assumed that the Euler angles and angular rates are known and that the transformation from the Euler angle rates to the body axis rates is desired. In this approach the previously derived rotation matrices for the Euler angles may be used. Then, the transformation actually wanted is the inverse of the transformation matrix thus derived.

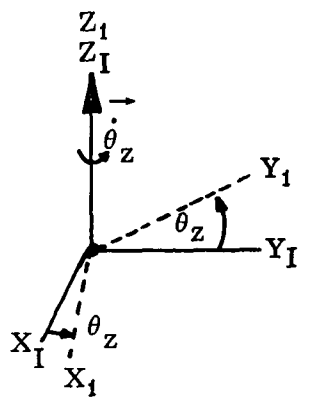
The rates of angular rotation about each of the Euler angle axes may be considered as vector quantities; i.e., because of vehicle rotation, there is some

component of vehicle angular velocity about each Euler angle axis. This vehicle angular rotation rate is represented mathematically as a set of vectors directed along the Euler angle axes, each with length proportional to the magnitude of the rotation rate about that axis.

If the magnitude of the vector along a particular Euler angle axis is known, the previously derived rotation matrices may be used to determine the corresponding components of vehicle angular rate about the body-fixed axes. The sum of the vehicle angular rate about each body-fixed axis caused by all three Euler angle rates is the total rotation rate of the vehicle about that axis. Figure 3 illustrates the transformation from Euler angle rates to vehicle angular rates and will be helpful in the derivation that follows.

Consider a rotation rate $\dot{\theta}_Z$ about the Z_I axis. This may be represented by a vector along the Z_I axis. To determine the components of this vector in the body-fixed coordinate system, it must be transformed into that reference frame. Making use of the first of the three rotation matrices:

$$\begin{pmatrix} \dot{\phi}_{x1} \\ \dot{\phi}_{y1} \\ \dot{\phi}_{z1} \end{pmatrix} = \begin{pmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_Z \end{pmatrix}$$

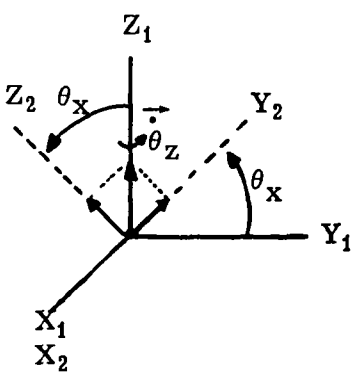


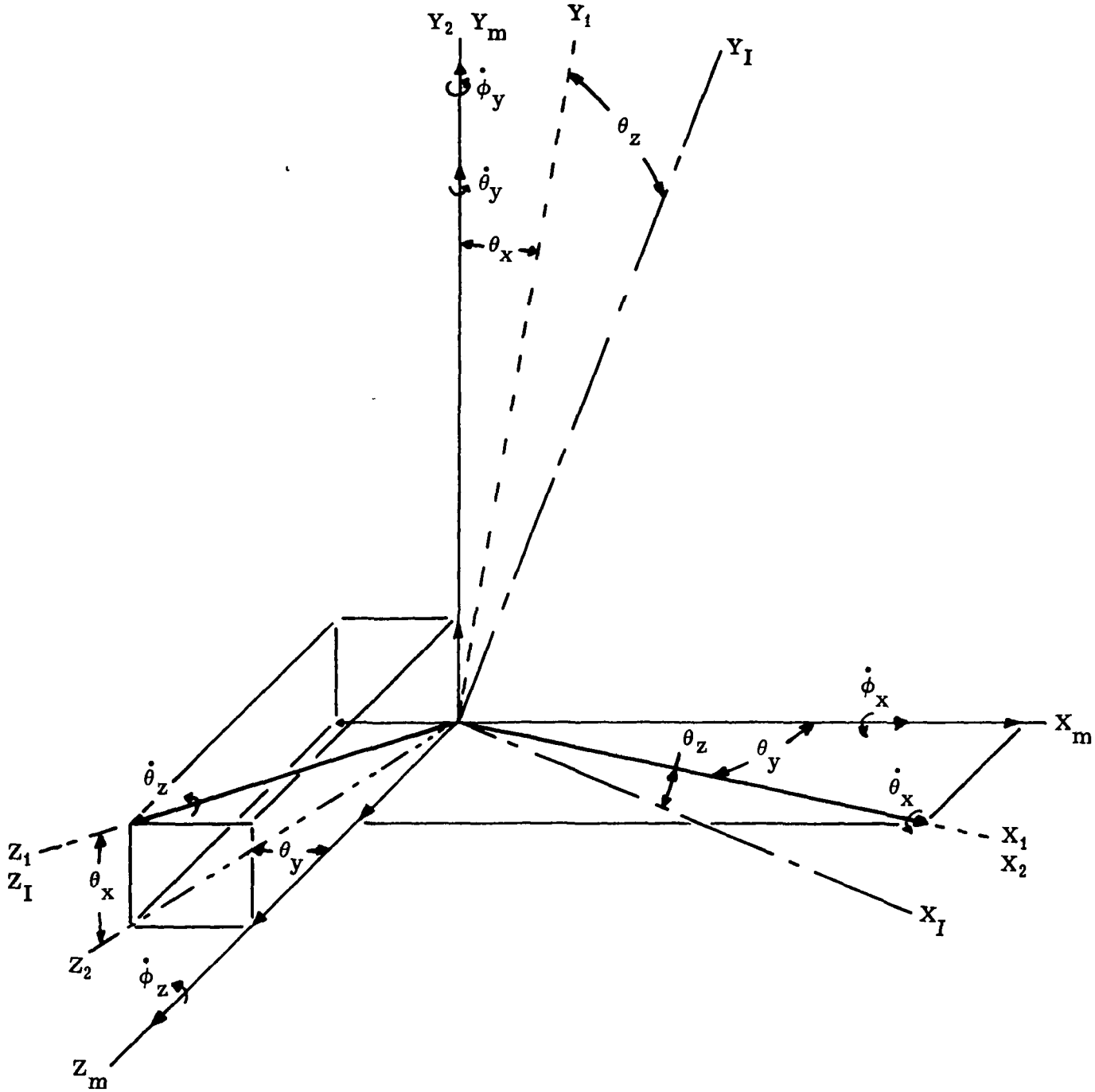
or

$$\dot{\phi}_{z1} = \dot{\theta}_Z, \quad \dot{\phi}_{x1} = \dot{\phi}_{y1} = 0.$$

Here, the rotation $\dot{\theta}_Z$ has a component only along the Z_1 axis. Next, the second rotation is applied.

$$\begin{pmatrix} \dot{\phi}_{x2} \\ \dot{\phi}_{y2} \\ \dot{\phi}_{z2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_Z \end{pmatrix}.$$





$$\begin{pmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \cos\theta_x \\ 0 & 1 & \sin\theta_x \\ \sin\theta_y & 0 & \cos\theta_y \cos\theta_x \end{pmatrix} \begin{pmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{pmatrix}$$

FIGURE 3. TRANSFORMATION FROM EULER ANGLE RATES TO VEHICLE ANGULAR RATES

or

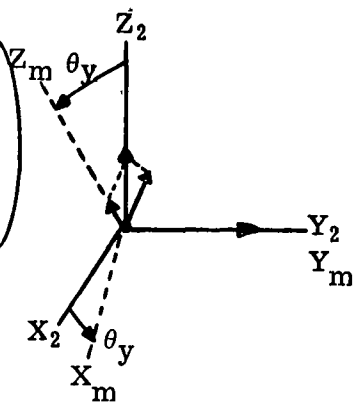
$$\dot{\phi}_{x2} = 0$$

$$\dot{\phi}_{y2} = \dot{\theta}_z \sin \theta_x$$

$$\dot{\phi}_{z2} = \dot{\theta}_z \cos \theta_x .$$

Applying the third rotation we obtain the components of $\vec{\dot{\theta}}_z$ in the body-fixed coordinate system.

$$\begin{pmatrix} \dot{\phi}_{xm} \\ \dot{\phi}_{ym} \\ \dot{\phi}_{zm} \end{pmatrix} = \begin{pmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta}_z \sin \theta_x \\ \dot{\theta}_z \cos \theta_x \end{pmatrix}$$



or

$$\dot{\phi}_{xm} = -\dot{\theta}_z \cos \theta_x \sin \theta_y$$

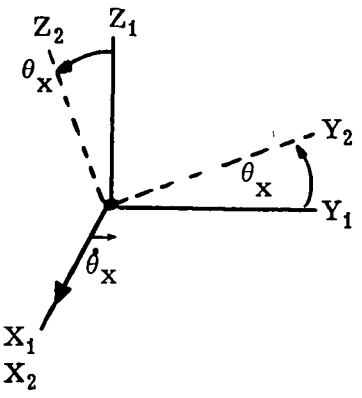
$$\dot{\phi}_{ym} = \dot{\theta}_z \sin \theta_x$$

$$\dot{\phi}_{zm} = \dot{\theta}_z \cos \theta_x \cos \theta_y .$$

The angular rates on the left hand side of the set of equations above represent the body-fixed axis rates corresponding to an Euler angle rate about only the Z_I axis. To these must be added the respective body-fixed axis rates caused by Euler angle rates $\dot{\theta}_x$ about the X_1 axis and $\dot{\theta}_y$ about the Y_2 axis.

Consider a rotation rate $\dot{\theta}_x$ about the X_1 axis represented by a vector directed along the X_1 axis. The components of this vector must be determined in the body-fixed coordinate system. First, rotate about X_1:

$$\begin{pmatrix} \dot{\phi}_{x_2} \\ \dot{\phi}_{y_2} \\ \dot{\phi}_{z_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} \dot{\theta}_x \\ 0 \\ 0 \end{pmatrix}$$

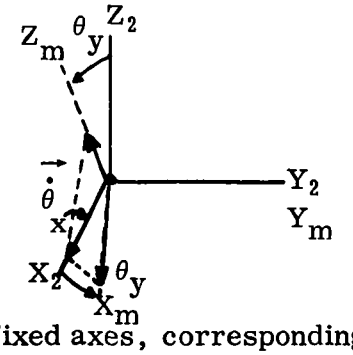


or

$$\dot{\phi}_{x2} = \dot{\theta}_x , \quad \dot{\phi}_{y2} = \dot{\phi}_{z2} = 0.$$

Next, rotate about Y_2 into the body-fixed coordinate system.

$$\begin{pmatrix} \dot{\phi}_{xm} \\ \dot{\phi}_{ym} \\ \dot{\phi}_{zm} \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} \dot{\theta}_x \\ 0 \\ 0 \end{pmatrix}.$$



The magnitude of the angular rates about the body-fixed axes, corresponding to an Euler angle rotation rate $\dot{\theta}_x$ is therefore given by:

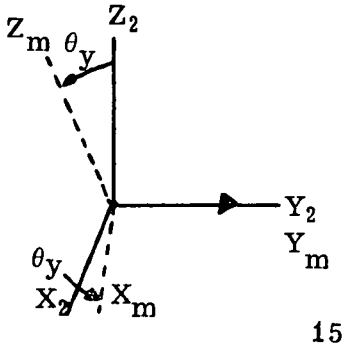
$$\dot{\phi}_{xm} = \dot{\theta}_x \cos\theta_y$$

$$\dot{\phi}_{ym} = 0$$

$$\dot{\phi}_{zm} = \dot{\theta}_x \sin\theta_y .$$

Consider a rotation rate $\dot{\theta}_y$ about the Y_2 axis represented by a vector along the Y_2 axis. This vector is transformed into the body-fixed coordinate system by the rotation.

$$\begin{pmatrix} \dot{\phi}_{xm} \\ \dot{\phi}_{ym} \\ \dot{\phi}_{zm} \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta}_y \\ 0 \end{pmatrix}$$



or

$$\dot{\phi}_{xm} = \dot{\phi}_{zm} = 0, \quad \dot{\phi}_{ym} = \dot{\theta}_y .$$

Adding the components of the respective vehicle angular rates caused by all the Euler angle rates, we derive the following matrix equation.

$$\begin{pmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \cos\theta_x \\ 0 & 1 & \sin\theta_x \\ \sin\theta_y & 0 & \cos\theta_y \cos\theta_x \end{pmatrix} \begin{pmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{pmatrix}$$

The value of the determinant of this transformation matrix is $\cos\theta_x$. Therefore, this transformation is not, in general, orthogonal.

Since the vehicle angular rates are known and the Euler angle rates are needed, they can be determined by taking the inverse of the transformation matrix above.

$$\begin{pmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{pmatrix} = \begin{pmatrix} \dfrac{\cos\theta_y \cos\theta_x}{\cos\theta_x} & 0 & \dfrac{\sin\theta_y \cos\theta_x}{\cos\theta_x} \\ \dfrac{\sin\theta_y \sin\theta_x}{\cos\theta_x} & \dfrac{\cos^2\theta_y \cos\theta_x + \sin^2\theta_y \cos\theta_x}{\cos\theta_x} & \dfrac{-\sin\theta_x \cos\theta_y}{\cos\theta_x} \\ \dfrac{-\sin\theta_y}{\cos\theta_x} & 0 & \dfrac{\cos\theta_y}{\cos\theta_x} \end{pmatrix} \begin{pmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix} .$$

It is obvious from this new transformation matrix that if $\cos\theta_x = 0$ the Euler angle rates are indeterminant. This singular point is the biggest disadvantage of the Euler angle approach. The sequence of rotations was chosen such that the singular point would be in θ_x since during the normal mission of Saturn V a 90 degree yaw maneuver will not be required in the boost phase of flight.

In simplest terms, the previous transformation becomes:

$$\begin{pmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ \sin\theta_y \tan\theta_x & 1 & -\cos\theta_y \tan\theta_y \\ \dfrac{-\sin\theta_y}{\cos\theta_x} & 0 & \dfrac{\cos\theta_y}{\cos\theta_x} \end{pmatrix} \begin{pmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix}.$$

To use the Euler angle method for determining the C matrix, we must continuously calculate the rate of change of the Euler angles and integrate to determine the present Euler angle positions. This requires only three integrations and is seen to be an advantage over the direction cosine method. The necessary calculation of sines and cosines in this method, however, is probably as time consuming for the computer as the additional integrations in the direction cosine method.

SECTION VI. REDUNDANT, OR FOUR EULER ANGLE, METHOD

One way to avoid the singular point in the previously described Euler angle method is to implement a redundant fourth angle. This angle will come last in the sequence from space-fixed to vehicle-fixed coordinate systems. It will be an unlimited rotation about X_I, called θ_{ox}, while θ_x will become a limited angle, e.g., 30°, and subsequently called θ_{xl}. The Euler angle rate $\dot{\theta}_{ox}$ will be calculated as necessary to insure that θ_{xl} stays well within its limited range. Therefore, the magnitude of the rate of θ_{ox} will be known from some calculations outside this area, e.g., in the guidance equations. Therefore, C now becomes:

$$C = \begin{pmatrix} \text{C MATRIX} \\ \text{FOR THREE} \\ \text{EULER ANGLE CASE} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{ox} & \sin\theta_{ox} \\ 0 & -\sin\theta_{ox} & \cos\theta_{ox} \end{pmatrix}.$$

To write out the new matrix C, let

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

then

$$c_{11} = \cos\theta_z \cos\theta_y - \sin\theta_z \sin\theta_{xl} \sin\theta_y$$

$$c_{12} = -\sin\theta_z \cos\theta_{xl} \cos\theta_{ox} + \cos\theta_z \sin\theta_y \sin\theta_{ox} + \sin\theta_z \sin\theta_{xl} \cos\theta_y \sin\theta_{ox}$$

$$c_{13} = \sin\theta_z \cos\theta_{xl} \sin\theta_{ox} + \cos\theta_z \sin\theta_y \cos\theta_{ox} + \sin\theta_z \sin\theta_{xl} \cos\theta_y \cos\theta_{ox}$$

$$c_{21} = \sin\theta_z \cos\theta_y + \cos\theta_z \sin\theta_{xl} \sin\theta_y$$

$$c_{22} = \cos\theta_z \cos\theta_{xl} \cos\theta_{ox} + \sin\theta_z \sin\theta_y \sin\theta_{ox} - \cos\theta_z \sin\theta_{xl} \cos\theta_y \sin\theta_{ox}$$

$$c_{23} = -\cos\theta_z \cos\theta_{xl} \sin\theta_{ox} + \sin\theta_z \sin\theta_y \cos\theta_{ox} - \cos\theta_z \sin\theta_{xl} \cos\theta_y \cos\theta_{ox}$$

$$c_{31} = -\cos\theta_{xl} \sin\theta_y$$

$$c_{32} = \sin\theta_{xl} \cos\theta_{ox} + \cos\theta_{xl} \cos\theta_y \sin\theta_{ox}$$

$$c_{33} = -\sin\theta_{xl} \sin\theta_{ox} + \cos\theta_{xl} \cos\theta_y \cos\theta_{ox}.$$

Next, the matrix which relates the four Euler angle rates and the vehicle body rates must be derived in the same manner as for the three Euler angle case. The components of the vehicle angular rates caused by $\dot{\theta}_z$ are given by the expression:

$$\begin{pmatrix} \dot{\phi}_{x1} \\ \dot{\phi}_{y1} \\ \dot{\phi}_{z1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{ox} & \sin\theta_{ox} \\ 0 & -\sin\theta_{ox} & \cos\theta_{ox} \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{xl} & \sin\theta_{xl} \\ 0 & -\sin\theta_{xl} & \cos\theta_{xl} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{pmatrix}$$

which reduces to:

$$\begin{pmatrix} \dot{\phi}_{x1} \\ \dot{\phi}_{y1} \\ \dot{\phi}_{z1} \end{pmatrix} = \begin{pmatrix} -\dot{\theta}_z \sin\theta_y \cos\theta_{xl} \\ \dot{\theta}_z (\sin\theta_{xl}\cos\theta_{ox} + \cos\theta_y \cos\theta_{xl}\sin\theta_{ox} \\ \dot{\theta}_z (-\sin\theta_{xl}\sin\theta_{ox} + \cos\theta_y \cos\theta_{xl}\cos\theta_{ox} \end{pmatrix}.$$

Similarly, the components of the vehicle angular rates caused by $\dot{\theta}_{xl}$ are given by:

$$\begin{pmatrix} \dot{\phi}_{x2} \\ \dot{\phi}_{y2} \\ \dot{\phi}_{z2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{ox} & \sin\theta_{ox} \\ 0 & -\sin\theta_{ox} & \cos\theta_{ox} \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} \dot{\theta}_{xl} \\ 0 \\ 0 \end{pmatrix}$$

which reduces to·

$$\begin{pmatrix} \dot{\phi}_{x2} \\ \dot{\phi}_{y2} \\ \dot{\phi}_{z2} \end{pmatrix} = \begin{pmatrix} \dot{\theta}_{xl}\cos\theta_y \\ \dot{\theta}_{xl}\sin\theta_{ox}\sin\theta_y \\ \dot{\theta}_{xl}\cos\theta_{ox}\sin\theta_y \end{pmatrix}.$$

The components of the vehicle angular rates caused by $\dot{\theta}_y$ are given by:

$$\begin{pmatrix} \dot{\phi}_{x3} \\ \dot{\phi}_{y3} \\ \dot{\phi}_{z3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{ox} & \sin\theta_{ox} \\ 0 & -\sin\theta_{ox} & \cos\theta_{ox} \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta}_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{\theta}_y \cos\theta_{ox} \\ -\dot{\theta}_y \sin\theta_{ox} \end{pmatrix}.$$

The components of the vehicle angular rates caused by $\dot{\theta}_{ox}$ are given by:

$$\begin{pmatrix} \dot{\phi}_{x4} \\ \dot{\phi}_{y4} \\ \dot{\phi}_{z4} \end{pmatrix} = \begin{pmatrix} \dot{\theta}_{ox} \\ 0 \\ 0 \end{pmatrix}.$$

Adding the components of the vehicle angular rates caused by all the Euler angle rates gives:

$$\begin{pmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cos\theta_y & -\sin\theta_y \cos\theta_{xl} \\ 0 & \cos\theta_{ox} & \sin\theta_{ox}\sin\theta_y & (\sin\theta_{xl}\cos\theta_{ox} + \cos\theta_y \cos\theta_{xl}\sin\theta_{ox}) \\ 0 & -\sin\theta_{ox} & \cos\theta_{ox}\sin\theta_y & (\cos\theta_y \cos\theta_{xl}\cos\theta_{ox} - \sin\theta_{xl}\sin\theta_{ox}) \end{pmatrix} \begin{pmatrix} \dot{\theta}_{ox} \\ \dot{\theta}_y \\ \dot{\theta}_{xl} \\ \dot{\theta}_z \end{pmatrix}.$$

This may be rewritten:

$$\begin{pmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix} = \begin{pmatrix} \dot{\theta}_{ox} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \cos\theta_y & -\sin\theta_y \cos\theta_{xl} \\ \cos\theta_{ox} & \sin\theta_{ox}\sin\theta_y & (\sin\theta_{xl}\cos\theta_{ox} + \cos\theta_y \cos\theta_{xl}\sin\theta_{ox}) \\ -\sin\theta_{ox} & \cos\theta_{ox}\sin\theta_y & (\cos\theta_y \cos\theta_{xl}\cos\theta_{ox} - \sin\theta_{xl}\sin\theta_{ox}) \end{pmatrix} \begin{pmatrix} \dot{\theta}_y \\ \dot{\theta}_{xl} \\ \dot{\theta}_z \end{pmatrix}.$$

or alternately:

$$\begin{pmatrix} \dot{\phi}_x - \dot{\theta}_{ox} \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \cos\theta_{xl} \\ \sin\theta_{ox}\sin\theta_y & \cos\theta_{ox} & (\sin\theta_{xl}\cos\theta_{ox} + \cos\theta_y \cos\theta_{xl}\cos\theta_{ox}) \\ \cos\theta_{ox}\sin\theta_y & -\sin\theta_{ox} & (\cos\theta_y\cos\theta_{xl}\cos\theta_{ox} - \sin\theta_{xl}\sin\theta_{ox}) \end{pmatrix} \begin{pmatrix} \theta_{xl} \\ \dot{\theta}_y \\ \dot{\theta}_z \end{pmatrix}.$$

To determine the rate of change of the Euler angles, given the vehicle angular rates, the inverse of this transformation matrix must be determined. The first step in calculating the inverse is to evaluate the determinant This will tell if the matrix is singular and if it meets a necessary test for orthogonality. Evaluating the determinant by multiplying out, combining terms, and applying trigonometric identities shows that the determinant is equal to cos θ_{xl}. Therefore, the transformation matrix is, in general, nonsingular and non-orthogonal,as expected. We proceed to determine the inverse.

Replacing each element in the transformation matrix by its cofactor results in the following matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = \cos^2\theta_{ox}\cos\theta_y\cos\theta_{xl} - \sin\theta_{xl}\sin\theta_{ox}\cos\theta_{ox} + \sin\theta_{xl}\sin\theta_{ox}\cos\theta_{ox} + \sin^2\theta_{ox}\cos\theta_y\cos\theta_{xl}$$

$$a_{12} = -\sin\theta_{ox}\cos\theta_{ox}\sin\theta_y\cos\theta_y\cos\theta_{xl} + \sin^2\theta_{ox}\sin\theta_y\sin\theta_{xl} + \cos^2\theta_{ox}\sin\theta_y\sin\theta_{xl} + \sin\theta_{ox}\cos\theta_{ox}\sin\theta_y\cos\theta_y\cos\theta_{xl}$$

$$a_{13} = -\sin^2 \theta_{ox} \sin \theta_y - \cos^2 \theta_{ox} \sin \theta_y$$

$$a_{21} = \sin \theta_{ox} \sin \theta_y \cos \theta_{xl}$$

$$a_{22} = \cos^2 \theta_y \cos \theta_{xl} \cos \theta_{ox} - \cos \theta_y \sin \theta_{xl} \sin \theta_{ox} + \sin^2 \theta_y \cos \theta_{xl} \cos^2 \theta_{ox}$$

$$a_{23} = \cos \theta_y \sin \theta_{ox}$$

$$a_{31} = + \sin \theta_y \cos \theta_{xl} \cos \theta_{ox}$$

$$a_{32} = -\cos \theta_y \sin \theta_{xl} \cos \theta_{ox} - \cos^2 \theta_y \cos \theta_{xl} \sin \theta_{ox} - \sin^2 \theta_y \cos \theta_{xl} \sin \theta_{ox}$$

$$a_{33} = \cos \theta_y \cos \theta_{ox}$$

which reduces to:

$$A = \begin{pmatrix} \cos \theta_y \cos \theta_{xl} & \sin \theta_y \sin \theta_{xl} & -\sin \theta_y \\ \sin \theta_y \cos \theta_{xl} \sin \theta_{ox} & -\cos \theta_y \sin \theta_{xl} \sin \theta_{ox} + \cos \theta_{xl} \cos \theta_{ox} & \cos \theta_y \sin \theta_{ox} \\ \sin \theta_y \cos \theta_{xl} \cos \theta_{ox} & -\cos \theta_y \sin \theta_{xl} \cos \theta_{ox} - \cos \theta_{xl} \sin \theta_{ox} & \cos \theta_y \cos \theta_{ox} \end{pmatrix}.$$

Taking the transpose of this matrix gives:

$$A^t = \begin{pmatrix} \cos\theta_y \cos\theta_{xl} & \sin\theta_y \cos\theta_{xl} \sin\theta_{ox} & \sin\theta_y \cos\theta_{xl} \cos\theta_{ox} \\ \sin\theta_y \sin\theta_{xl} & -\cos\theta_y \sin\theta_{xl} \sin\theta_{ox} + \cos\theta_{xl} \cos\theta_{ox} & -\cos\theta_y \sin\theta_{xl} \cos\theta_{ox} - \cos\theta_{xl} \sin\theta_{ox} \\ -\sin\theta_y & \cos\theta_y \sin\theta_{ox} & \cos\theta_y \cos\theta_{ox} \end{pmatrix}.$$

Dividing each element by the value of the determinant gives the inverse B which we use as follows:

$$\begin{pmatrix} \dot{\theta}_{xl} \\ \dot{\theta}_y \\ \dot{\theta}_z \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} \dot{\phi}_x - \dot{\theta}_{ox} \\ \dot{\phi}_y \\ \dot{\phi}_z \end{pmatrix}$$

where

$$b_{11} = \cos\theta_y$$

$$b_{12} = \sin\theta_y \sin\theta_{ox}$$

$$b_{13} = \sin\theta_y \cos\theta_{ox}$$

$$b_{21} = \sin\theta_y \tan\theta_{xl}$$

$$b_{22} = -\cos\theta_y \tan\theta_{xl} \sin\theta_{ox} + \cos\theta_{ox}$$

$$b_{23} = -\cos\theta_y \tan\theta_{xl} \cos\theta_{ox} - \sin\theta_{ox}$$

$$b_{31} = -\sin\theta_y / \cos\theta_{xl}$$

$$b_{32} = \cos \theta_y \sin \theta_{ox} / \cos \theta_{xl}$$

$$b_{33} = \frac{\cos \theta_y \cos \theta_{ox}}{\cos \theta_{xl}} .$$

Thus, from the angular rates of the vehicle, the angular rates of the Euler angles may be determined using this matrix. Integrating the Euler angle rates continuously during flight keeps the Euler angles updated. This requires four integrations now instead of three as with the previous method. The calculations are more complex in this modification of the basic Euler angle method but the problem of the singular point is avoided.

SECTION VII. FOUR PARAMETER METHOD

This method of calculating the transformation matrix is also known as the quaternion method and the Cayley-Klein method. The four parameters avoid the necessity of calculating the sines and cosines as in the Euler angle method. Also, it has no singular point as do the three Euler angles. Four integrations are necessary, which is the minimum number for a transformation with no singular point.

The transformation matrix is derived in some detail in the following formulations of the four parameter method. By means of quaternions, Cayley-Klein parameters, and Euler's theorem, the same transformation matrix is derived. The relation of the body rates to the four parameter rates is also derived.

There are other ways of implementing the calculation of the C matrix using four parameters; however, the method presented here seems to be the most straightforward and practical for solution in an onboard digital computer.

A. FORMULATIONS OF THE TRANSFORMATION MATRIX

1. Euler's Theorem

Any real rotation may be expressed as a rotation through some angle about some fixed axis.

Proof See Appendix C.

2. Application of Euler's Theorem to Rotations

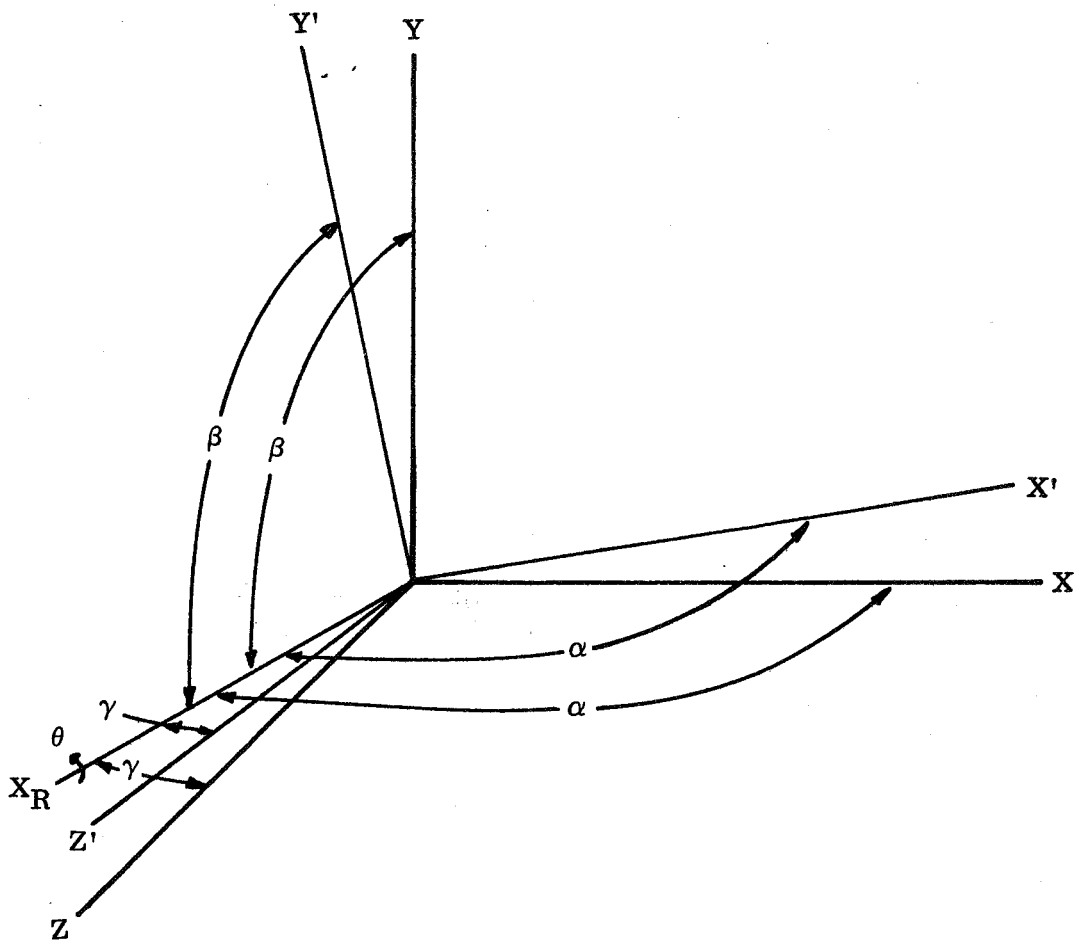
Consider two coordinate systems X, Y, Z and X', Y', Z'. Assume X, Y, Z to be fixed in inertial space and X', Y', Z' free to move in an arbitrary manner. Both systems have a common origin. Assume that initially the two coordinate systems are coincident. Then the X', Y', Z' system is rotated arbitrarily. By Euler's theorem this rotation can be expressed as a rotation through an angle θ about an axis which has direction angles α, β, γ, respectively, with the axes of the X, Y, Z system. Note that also by Euler's theorem this axis of rotation has the same direction angles with the axes of the X', Y', Z' system.

To find an expression for the orthogonal transformation matrix in terms of θ, α, β, and γ, first, define a third coordinate system X_R, Y_R, Z_R fixed with respect to X, Y, Z, with X_R along the axis of rotation, Y_R in the XY plane, and Z_R orthogonal in a right-handed coordinate system. Now the rotation θ is a rotation through the angle θ about the X_R axis (Fig. 4). The rotation of the X', Y', Z' coordinate system through the angle θ may be considered as the resultant of three separate rotations: first, a rotation of the X', Y', Z' system into coincidence with the X_R, Y_R, Z_R system, second, a rotation θ about X_R; third, an inverse of the first rotation to restore the original relationship between the X', Y', Z' and X_R, Y_R, Z_R coordinate systems. Each of these rotations will be described by an orthogonal transformation matrix. Then the three matrices will be multiplied together to derive the complete transformation.

Consider the first transformation:

$$\begin{pmatrix} X_R \\ Y_R \\ Z_R \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}. \tag{1}$$

The problem now is to find the elements of this matrix to describe this rotation. Since X_R makes angles of α, β, γ with X', Y', Z', respectively, then $a_{11} = \cos \alpha$; $a_{12} = \cos \beta$; $a_{13} = \cos \gamma$. Since Y_R lies in the X' Y' plane, a_{23} must be 0. With this information we know



$$e_1 \equiv \cos \frac{\theta}{2}$$

$$e_2 \equiv \cos \gamma \sin \frac{\theta}{2}$$

$$e_3 \equiv \cos \beta \sin \frac{\theta}{2}$$

$$e_4 \equiv \cos \alpha \sin \frac{\theta}{2}$$

FIGURE 4. FOUR PARAMETERS

$$A = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2}$$

and since A is an orthogonal transformation matrix·

$$A \cdot A^t = I \text{ and } A^t \cdot A = I \tag{3}$$

which can be written out as·

$$\begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \cos\alpha & a_{21} & a_{31} \\ \cos\beta & a_{22} & a_{32} \\ \cos\gamma & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

and

$$\begin{pmatrix} \cos\alpha & a_{21} & a_{31} \\ \cos\beta & a_{22} & a_{32} \\ \cos\gamma & 0 & a_{33} \end{pmatrix} \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

From equation 4:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{6}$$

$$a_{21}\cos\alpha + a_{22}\cos\beta = 0 \tag{7}$$

$$a_{21}^2 + a_{22}^2 = 1 \tag{8}$$

$$\therefore\ a_{22} = \sqrt{1 - a_{21}^2}$$

and by substituting equation 8 into equation 7:

$$a_{21} \cos \alpha + \sqrt{1-a_{21}^2} \cos \beta = 0$$

$$a_{21} \cos \alpha = - \sqrt{1-a_{21}^2} \cos \beta$$

$$a_{21}^2 \cos^2 \alpha = (1-a_{21}^2) \cos^2 \beta$$

$$a_{21}^2 \cos^2 \alpha = \cos^2 \beta - a_{21}^2 \cos^2 \beta$$

$$a_{21}^2 \cos^2 \alpha + a_{21}^2 \cos^2 \beta = \cos^2 \beta$$

$$a_{21}^2 (\cos^2 \alpha + \cos^2 \beta) = \cos^2 \beta$$

$$a_{21}^2 (1 - \cos^2 \gamma) = \cos^2 \beta \qquad \text{from equation 6}$$

$$a_{21}^2 = \frac{\cos^2 \beta}{\sin^2 \gamma} \qquad (9)$$

$$\underline{a_{21} = \pm \cos \beta \operatorname{osc} \gamma} \qquad (10)$$

$$a_{22}^2 = 1 - a_{21}^2 \qquad \text{from equation 8}$$

$$a_{22}^2 = 1 - \cos^2 \beta \csc^2 \gamma$$

$$a_{22}^2 = \frac{\csc^2 \gamma}{\csc^2 \gamma} - \cos^2 \beta \csc^2 \gamma$$

$$a_{22}^2 = \csc^2 \gamma (\sin^2 \gamma - \cos^2 \beta)$$

$$a_{22}^2 = \csc^2 \gamma (1 - \cos^2 \gamma - \cos^2 \beta)$$

$$a_{22}^2 = \csc^2 \gamma \cos^2 \alpha \qquad \text{from equation 6} \qquad (11)$$

$$\underline{a_{22} = \pm \csc \gamma \cos \alpha} \qquad (12)$$

$$\cos^2 \gamma + a^2_{33} = 1 \qquad \text{from equation 5}$$

$$a^2_{33} = 1 - \cos^2 \gamma \qquad (13)$$

$$a^2_{33} = \sin^2 \gamma \qquad (14)$$

$$\underline{a_{33} = \pm \sin \gamma}$$

from equation 5

$$\cos \beta \cos \gamma + a_{32}\, a_{33} = 0$$

$$a_{32}\, a_{33} = - \cos \beta \cos \gamma$$

$$a_{32} = \frac{-\cos \beta \cos \gamma}{a_{33}}$$

$$a_{32} = \pm \frac{\cos \beta \cos \gamma}{\sin \gamma}$$

$$\underline{a_{32} = \pm \cos \beta \cot \gamma} \qquad (15)$$

from equation 4

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$$

$$a_{31}^2 + \cos^2 \beta \cot^2 \gamma + \sin^2 \gamma = 1$$

$$a_{31}^2 = 1 - \sin^2 \gamma - \cos^2 \beta \cot^2 \gamma$$

$$a_{31}^2 = \cos^2 \gamma - \cos^2 \beta \cot^2 \gamma$$

$$a_{31}^2 = \frac{\cos^2 \gamma \sin^2 \gamma}{\sin^2 \gamma} - \cos^2 \beta \cot^2 \gamma$$

$$a_{31}^2 = \cot^2 \gamma \sin^2 \gamma - \cos^2 \beta \cot^2 \gamma$$

$$a_{31}^2 = \cot^2 \gamma \ (\sin^2 \gamma - \cos^2 \beta)$$

$$a_{31}^2 = \cot^2 \gamma \ (1 - \cos^2 \gamma - \cos^2 \beta)$$

$$a_{31}^2 = \cot^2 \gamma \cos^2 \alpha \tag{16}$$

$$\underline{a_{31} = \pm \cot \gamma \cos \alpha}. \tag{17}$$

All elements of this matrix have now been determined except for sign; therefore,

$$A = \begin{pmatrix} \cos \alpha & \cos \beta & \pm \cos \gamma \\ \pm \cos \beta \csc \gamma & \pm \csc \gamma \cos \alpha & 0 \\ \pm \cot \gamma \cos \alpha & \pm \cos \beta \cot \gamma & \sin \gamma \end{pmatrix}. \tag{18}$$

To determine signs for the ambiguous elements of A, take the case of $\alpha = 0$. Then by the construction of the X_R, Y_R, Z_R coordinate system and the definitions of angles α, β, γ; $\beta = 90°$ and $\gamma = 90°$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Therefore, elements a_{22} and a_{33} must be positive. Going back to the relations given in equation 3 we can determine the signs of the remaining elements.

$$AA^t = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tag{20}$$

$$\begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \pm\cos\beta\csc\gamma & \csc\gamma\cos\alpha & 0 \\ \pm\cot\gamma\cos\alpha & \pm\cos\beta\cot\gamma & \sin\gamma \end{pmatrix} \begin{pmatrix} \cos\alpha & \pm\cos\beta\csc\gamma & \pm\cot\gamma\cos\alpha \\ \cos\beta & \csc\gamma\cos\alpha & \pm\cos\beta\cot\gamma \\ \cos\gamma & 0 & \sin\gamma \end{pmatrix}$$

$$A^tA = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tag{21}$$

$$\begin{pmatrix} \cos\alpha & \pm\cos\beta\csc\gamma & \pm\cot\gamma\cos\alpha \\ \cos\beta & \csc\gamma\cos\alpha & \pm\cos\beta\cot\gamma \\ \cos\gamma & 0 & \sin\gamma \end{pmatrix} \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \pm\cos\beta\csc\gamma & \csc\gamma\cos\alpha & 0 \\ \pm\cot\gamma\cos\alpha & \pm\cos\beta\cot\gamma & \sin\gamma \end{pmatrix}.$$

Since, $\pm\cos\beta\csc\gamma\cos\alpha + \csc\gamma\cos\alpha\cos\beta = 0$ (from equation 20) (22)

element a_{21} must be negative.

Since, $\cos\alpha\cos\gamma \pm\cot\gamma\cos\alpha\sin\gamma = 0$ (from equation 21) (23)

$$\cos\alpha\cos\gamma \pm \frac{\cos\gamma}{\sin\gamma}\cdot\sin\gamma\cos\alpha = 0$$

therefore, element a_{31} must be negative.

Since, $\cos\gamma\cos\beta \pm \cos\beta\cot\gamma\sin\gamma = 0$ (from equation 21) (24)

$$\cos\gamma\cos\beta \pm \cos\beta\frac{\cos\gamma}{\sin\gamma}\sin\gamma = 0$$

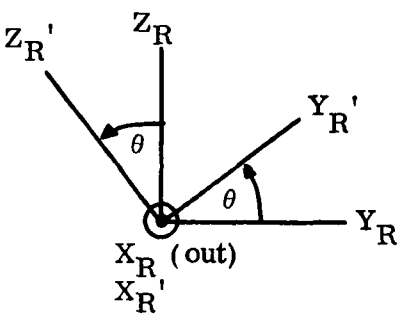
therefore, element a_{32} must be negative.

Now, A is completely determined.

$$A = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ -\cos\beta\cos\gamma & \csc\gamma\cos\alpha & 0 \\ -\cot\gamma\cos\alpha & -\cos\beta\cot\gamma & \sin\gamma \end{pmatrix}. \tag{25}$$

Next consider the second transformation, a rotation through the angle θ about the X_R axis:

$$\begin{pmatrix} X_R' \\ Y_R' \\ Z_R' \end{pmatrix} = R \begin{pmatrix} X_R \\ Y_R \\ Z_R \end{pmatrix}. \tag{26}$$



Therefore,

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{27}$$

The third transformation is the inverse (or transpose in the particular case of orthogonal rotation matrices) of the first transformation.

$$A^{-1} = \begin{pmatrix} \cos\alpha & -\cos\beta\csc\gamma & -\cos\gamma\cos\alpha \\ \cos\beta & \csc\gamma\cos\alpha & -\cos\beta\cot\gamma \\ \cos\gamma & 0 & \sin\gamma \end{pmatrix}. \tag{28}$$

The complete transformation is then the product of these three transformations:

$$B = A^{-1} R A. \tag{29}$$

Carrying out the indicated matrix multiplication and making the following substitutions:

$$\begin{aligned} e_1 &\equiv \cos\frac{\theta}{2} \\ e_2 &\equiv \cos\gamma\sin\frac{\theta}{2} \\ e_3 &\equiv \cos\beta\sin\frac{\theta}{2} \\ e_4 &\equiv \cos\alpha\sin\frac{\theta}{2} \end{aligned} \tag{30}$$

give the following result.

$$B = \begin{pmatrix} e_1^2 - e_2^2 - e_3^2 + e_4^2 & 2(e_1 e_2 + e_3 e_4) & 2(e_2 e_4 - e_1 e_3) \\ 2(e_3 e_4 - e_1 e_2) & e_1^2 - e_2^2 + e_3^2 - e_4^2 & 2(e_1 e_4 + e_2 e_3) \\ 2(e_1 e_3 + e_2 e_4) & 2(e_2 e_3 - e_1 e_4) & e_1^2 + e_2^2 - e_3^2 - e_4^2 \end{pmatrix}. \tag{31}$$

We shall now look at two simpler methods of deriving B.

3. Transformation Matrix Derived Using Quaternions

Hamilton in 1843 developed a new concept called a quaternion. It is made up of four parts:

$$q = s + i\,a + j\,b + k\,c \tag{32}$$

where s, a, b, c are real numbers, and i, j, k are indices governed by the following rules·

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \quad i\,j = -j\,i = k \\ &j\,k = -k\,j = i, \quad k\,i = -i\,k = j. \end{aligned} \tag{33}$$

The conjugate of q is given by:

$$q^* = s - ia - jb - kc. \tag{34}$$

Therefore:

$$qq^* = s^2 + a^2 + b^2 + c^2 = q^*q. \tag{35}$$

Now qq^* is called the length or norm of the quaternion. A quaternion of length 1 is called a versor. The scalar part of q is called s and the rest of q is called the vector part.

Given an ordinary vector $\vec{V}$ in X, Y, Z we can write this vector in quaternion form as follows·

$$V = iX + jY + kZ \tag{36}$$

where the i, j, k obey the rules in equation 33.

Perform the operation:

$$q^*\ V\ q = V' \tag{37}$$

where q is a versor. Writing out equation 37 gives:

$$(s - ia - jb - kc)\ (iX + jY + kZ)\ (s + ia + jb + kc) = V'. \tag{38}$$

Expanding equation 38 and using the rules in equation 33 give the following:

$$\begin{aligned} V' = {} & iXs^2 - Xas + kXbs - jXcs + jYs^2 - kYas - Ybs + iYcs + kZs^2 \\ & + jZas - iZbs - Zcs + Xas + iXa^2 + jXab + kXac - kYas - jYa^2 \\ & + iYab + Yac + jZas - kZa^2 - Zab + iZac + kXbs + jXab - iXb^2 \\ & - Xbc + Ybs + iYab + jYb^2 + kYbc - iZbs + Zab - kZb^2 + jZbc \\ & - jXcs + kXac + Xbc - iXc^2 + iYcs - Yac + kYbc - jYc^2 + Zcs \\ & + iZac + jZbc + kZc^2 \end{aligned} \tag{39}$$

$$\begin{aligned} V' = {} & i[X(s^2 + a^2 - b^2 - c^2) + Y(cs + ab + ab + cs) + Z(-bs + ac - bs + ac)] \\ & + j[X(-cs + ab + ab - cs) + Y(s^2 - a^2 + b^2 - c^2) + Z(+as + as + bc + bc)] \\ & + k[X(bs + ac + bs + ac) + Y(-as - as + bc + bc) + Z(s^2 - a^2 - b^2 + c^2)]. \end{aligned} \tag{40}$$

V' can be expressed in vector form as follows, since it contains no scalar part.

$$\vec{V'} = \vec{i}X' + \vec{j}Y' + \vec{k}Z'. \tag{41}$$

From equations 40 and 41, the result can be written in matrix form.

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} s^2 + a^2 - b^2 - c^2 & 2(ab + cs) & 2(ac - bs) \\ 2(ab - cs) & s^2 - a^2 + b^2 - c^2 & 2(as + bc) \\ 2(ac + bc) & 2(bc - as) & s^2 - a^2 - b^2 + c^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{42}$$

Comparing equations 31 and 42 gives the following relationships:

$$e_1 = s, \quad e_2 = c, \quad e_3 = b, \quad e_4 = a. \tag{43}$$

With these substitutions we have derived, by means of quaternions, the same transformation as previously derived.

4. Transformation Matrix Derived Using Cayley-Klein Parameters

In this development use is made of a 2 x 2 complex matrix to represent a real rotation. For the complex matrix H,

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} , \tag{44}$$

the following requirements are made:

$$H^*H = I \tag{45}$$

$$|H| = 1 \tag{46}$$

where H^* is the complex conjugate of H^t and is called the adjoint of H.

Writing out equation 45 gives:

$$\begin{pmatrix} h_{11}^* & h_{21}^* \\ h_{12}^* & h_{22}^* \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{47}$$

Then multiplying out equation 47 gives:

$$\begin{aligned} h_{11}^* h_{11} + h_{21}^* h_{21} = 1, \quad h_{11}^* h_{12} + h_{21}^* h_{22} = 0 \\ h_{12}^* h_{11} + h_{22}^* h_{21} = 0, \quad h_{12}^* h_{12} + h_{22}^* h_{22} = 1. \end{aligned} \tag{48}$$

From equation 46,

$$h_{11}\, h_{22} - h_{21}\, h_{12} = 1 \,. \tag{49}$$

Comparing equation 49 and the first equation in equation 48 results in the following:

$$h_{11}^{*} = h_{22}, \quad h_{21}^{*} = -h_{12} \,. \tag{50}$$

From the last part of equation 50, equation 51 is derived.

$$\begin{aligned} h_{21}^{*} &= -h_{12} = -a - bi \\ h_{21} &= -h_{12}^{*} = -a + bi. \end{aligned} \tag{51}$$

From equations 50 and 51, matrix H can be written as follows:

$$H = \begin{pmatrix} h_{11} & h_{12} \\ -h_{12}^{*} & h_{11}^{*} \end{pmatrix}. \tag{52}$$

The elements of this matrix are usually referred to as the Cayley-Klein parameters. They may be expressed as follows:

$$\begin{aligned} h_{11} &= e_1 + e_2\, i \\ h_{12} &= e_3 + e_4\, i \\ -h_{12}^{*} &= -e_3 + e_4\, i \\ h_{11}^{*} &= e_1 - e_2\, i. \end{aligned} \tag{53}$$

Now we make use of another complex matrix P where:

$$P = \begin{pmatrix} Z & X - i\,Y \\ X + i\,Y & -Z \end{pmatrix}. \tag{54}$$

Note that P is Hermitian, i.e., $P = P^*$.

We wish to perform a transformation of P represented by the following:

$$P' = H\,P\,H^*. \tag{55}$$

This is a similarity transformation. Therefore, P' must have the same form as P. Thus

$$P' = \begin{pmatrix} Z' & X' - iY' \\ X' + iY' & -Z' \end{pmatrix}. \tag{56}$$

Writing out equation 55 gives:

$$\begin{pmatrix} Z' & X' - iY' \\ X' + iY' & -Z' \end{pmatrix} = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} \begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix} \begin{pmatrix} e_1 - ie_2 & -e_3 - ie_4 \\ e_3 - ie_4 & e_1 + ie_2 \end{pmatrix}. \tag{57}$$

Multiplying out equation 57 gives:

$$\begin{aligned} Z' = {} & [(e_1 + ie_2)\,Z + (e_3 + ie_4)\,(X + iY)]\,[e_1 - ie_2] \\ & + [(e_1 + ie_2)\,(x - iY) - Z\,(e_3 + ie_4)]\,[e_3 - ie_4] \end{aligned} \tag{58}$$

$$\begin{aligned} X' - iY' = {} & [(e_1 + ie_2)\,Z + (e_3 + ie_4)\,(X + iY)]\,[-e_3 - ie_4] \\ & + [(e_1 + ie_2)\,(X - iY) - Z\,(e_3 + ie_4)]\,[e_1 + ie_2]. \end{aligned} \tag{59}$$

Multiplying out equation 58 gives:

$$\begin{aligned} Z' = {} & (e_1 Z + ie_2 Z + e_3 X + ie_4 X + ie_3 Y - e_4 Y)\,(e_1 - ie_2) \\ & + (e_1 X - ie_1 Y + ie_2 X + e_2 Y - e_3 Z - ie_4 Z)\,(e_3 - ie_4) \end{aligned}$$

$$Z' = e_1^2 Z + ie_1e_2Z + e_1e_3X + ie_1e_4X + ie_1e_3Y - e_1e_4Y$$

$$- ie_1 e_2Z + e_2^2 Z - ie_2e_3X + e_2e_4X + e_2e_3Y + ie_2e_4Y$$

$$+ e_1e_3X - ie_1e_3Y + ie_2e_3X + e_2e_3Y - e_3^2 Z - ie_3e_4Z$$

$$- ie_1e_4X - e_1e_4Y + e_2e_4X - ie_2e_4Y + ie_3e_4Z - e_4^2 Z$$

$$Z' = (2\, e_1e_3 + 2\, e_2e_4)\ X + (2\, e_2e_3 - 2\, e_1e_4)\ Y + (e_1^2 + e_2^2 - e_3^2 - e_4^2)Z. \qquad (60)$$

Multiplying out equation 59 gives:

$$X' - iY' = (e_1Z + ie_2Z + e_3X + ie_4X + ie_3Y - e_4Y)\ (- e_3 - ie_4)$$

$$+ (e_1X - ie_1Y + ie_2X + e_2Y - e_3Z - ie_4Z)\ (e_1 + ie_2)$$

$$X' - iY' = - e_1e_3Z - ie_2e_3Z - e_3^2 X - ie_3e_4X - ie_3^2 Y + e_3e_4Y$$

$$- ie_1e_4Z + e_2e_4Z - ie_3e_4X + e_4^2 X + e_3e_4Y + ie_4^2 Y$$

$$+ e_1^2 X - ie_1^2 Y + ie_1e_2X + e_1e_2Y - e_1e_3Z - ie_1e_4Z$$

$$+ ie_1e_2X + e_1e_2Y - e_2^2 X + ie_2^2 Y - ie_2e_3Z + e_2e_4Z$$

$$X' - iY' = (e_1^2 - e_2^2 - e_3^2 + e_4^2)X + (2\, e_1e_2 + 2\, e_3e_4)\, Y$$

$$+ (2\, e_2e_4 - 2\, e_1e_3)\ Z + i(2\, e_1e_2 - 2\, e_3e_4)X$$

$$+ \ i(- e_1^2 + e_2^2 - e_3^2 + e_4^2)\, Y + i(- 2\, e_1e_4 - 2\, e_2e_3)\, Z.$$

Collecting real and imaginary terms:

$$X' = (e_1^2 - e_2^2 - e_3^2 + e_4^2)X + 2(e_1e_2 + e_3e_4)\, Y + 2(e_2e_4 - e_1e_3)\, Z \qquad (61)$$

$$Y' = 2(e_3e_4 - e_1e_2)\, X + (e_1^2 - e_2^2 + e_3^2 - e_4^2)\ Y + 2\,(e_1e_4 + e_2e_3)\, Z. \qquad (62)$$

Equations 60 , 61, and 62 may be collected and written in matrix notation to give the following·

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} e_1^2 - e_2^2 - e_3^2 + e_4^2 & 2(e_1e_2 + e_3e_4) & 2(e_2e_4 - e_1e_3) \\ 2(e_3e_4 - e_1e_2) & e_1^2 - e_2^2 + e_3^2 - e_4^2 & 2(e_1e_4 + e_2e_3) \\ 2(e_1e_3 + e_2e_4) & 2(e_2e_3 - e_1e_4) & e_1^2 + e_2^2 - e_3^2 - e_4^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{63}$$

Thus, we have derived by means of the Cayley-Klein parameters the same transformation matrix previously derived by quaternions (eq. 42) and Euler's theorem (eq. 31).

B. A METHOD OF CALCULATING THE FOUR PARAMETERS ASSUMING INFINITESIMAL ROTATIONS

Let us recall the H matrix developed using Cayley-Klein parameters:

$$H = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix}. \tag{64}$$

Substituting the angular expressions for the e's from equation 30 :

$$H = \begin{pmatrix} \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\gamma & \sin\frac{\theta}{2}\cos\beta + i\sin\frac{\theta}{2}\cos\alpha \\ -\sin\frac{\theta}{2}\cos\beta + i\sin\frac{\theta}{2}\cos\alpha & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\cos\gamma \end{pmatrix}. \tag{65}$$

If we assume that we have an infinitesimal rotation, then $\sin\frac{\theta}{2} = \frac{\Delta\theta}{2}$, $\cos\frac{\Delta\theta}{2} = 1$, and this infinitesimal rotation can be expressed as:

$$\Delta H = \begin{pmatrix} 1 + i\frac{\Delta\theta}{2}\cos\gamma & \frac{\Delta\theta}{2}\cos\beta + i\frac{\Delta\theta}{2}\cos\alpha \\ -\frac{\Delta\theta}{2}\cos\beta + i\frac{\Delta\theta}{2}\cos\alpha & 1 - i\frac{\Delta\theta}{2}\cos\gamma \end{pmatrix}. \tag{66}$$

Matrix ΔH may be expressed as the sum of two matrices:

$$\Delta H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta\theta}{2} \begin{pmatrix} i\cos\gamma & \cos\beta + i\cos\alpha \\ -\cos\beta + i\cos\alpha & -i\cos\gamma \end{pmatrix}. \tag{67}$$

This may be written:

$\Delta H = I + E$, where I is the 2 x 2 identity matrix

Next, the assumption is made that the infinitesimal rotation ΔH takes place during some small time interval Δt. If H is the transformation matrix prior to the infinitesimal rotation and H' is the transformation matrix after the infinitesimal rotation:

$$\frac{dH}{dt} = \lim_{\Delta t \to o} \frac{H' - H}{\Delta t}. \tag{68}$$

Assume that H' is the resultant of two rotations; first, a rotation H and then the infinitesimal rotation ΔH. Thus

$$\frac{dH}{dt} = \lim_{\Delta t \to o} \frac{\Delta H \cdot H - H}{\Delta t} \tag{69}$$

$$\frac{dH}{dt} = \lim_{\Delta t \to o} \frac{(\Delta H - I)H}{\Delta t}. \tag{70}$$

I is the identity matrix, such that $IH = H$.

$$\frac{dH}{dt} = \lim_{\Delta t \to o} \frac{E}{\Delta t} \cdot H. \tag{71}$$

Equation 71 makes use of the relationship stated in equation 67. The limiting process is applicable only to E; therefore, equation 71 can be written out as follows:

$$\frac{dH}{dt} = \lim_{\Delta t \to o} \frac{\Delta\theta}{2\Delta t} \begin{pmatrix} i\cos\gamma & \cos\beta + i\cos\alpha \\ -\cos\beta + i\cos\alpha & -i\cos\gamma \end{pmatrix} H, \qquad (72)$$

The $\lim_{\Delta t \to o} \frac{\Delta\theta}{\Delta t}$ is the magnitude of the angular velocity vector $\frac{d\theta}{dt}$.

Define $\dot{\phi}_x$, $\dot{\phi}_y$, $\dot{\phi}_z$ as the components of angular velocity along the X, Y, and Z axes, respectively. Then we have the following:

$$\dot{\phi}_x = \frac{d\theta}{dt}\cos\alpha, \quad \dot{\phi}_y = \frac{d\theta}{dt}\cos\beta, \quad \dot{\phi}_z = \frac{d\theta}{dt}\cos\gamma. \qquad (73)$$

Using these relationships and definitions in equation 72 gives:

$$\frac{dH}{dt} = \frac{1}{2} \begin{pmatrix} i\dot{\phi}_z & \dot{\phi}_y + i\dot{\phi}_x \\ -\dot{\phi}_y + i\dot{\phi}_x & -i\dot{\phi}_z \end{pmatrix} \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix}. \qquad (74)$$

Equation 74 may be expressed as follows, since the time derivative of H is just a matrix with elements equal to the time derivatives of the elements of H:

$$\begin{pmatrix} \dot{e}_1 + i\dot{e}_2 & \dot{e}_3 + i\dot{e}_4 \\ -\dot{e}_3 + i\dot{e}_4 & \dot{e}_1 - i\dot{e}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i\dot{\phi}_z & \dot{\phi}_y + \dot{\phi}_x \\ -\dot{\phi}_y + i\dot{\phi} & -i\dot{\phi}_z \end{pmatrix} \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix}. \qquad (75)$$

Multiplying out and solving for the $\dot{e}$'s give:

$$\dot{e}_1 + i\dot{e}_2 = \frac{1}{2}\left[ie_1\dot{\phi}_z - e_2\dot{\phi}_z - e_3\dot{\phi}_y - ie_4\dot{\phi}_x + ie_4\dot{\phi}_y - ie_3\dot{\phi}_x\right]$$

$$\dot{e}_3 + i\dot{e}_4 = \frac{1}{2}\left[ie_3\dot{\phi}_z - e_4\dot{\phi}_z + e_1\dot{\phi}_y + ie_1\dot{\phi}_x - ie_2\dot{\phi}_y + e_2\dot{\phi}_x\right]$$

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} -e_4 & -e_3 & -e_2 \\ -e_3 & +e_4 & +e_1 \\ +e_2 & +e_1 & -e_4 \\ +e_1 & -e_2 & +e_3 \end{pmatrix} \begin{pmatrix} \dot{\phi}_x/2 \\ \dot{\phi}_y/2 \\ \dot{\phi}_z/2 \end{pmatrix}. \tag{76}$$

This matrix would be used to calculate the $\dot{e}$'s in an actual implementation. The $\dot{e}$'s must be integrated to give the e's in the implementation. From equation 30, we can derive a basic equation which may be used to preserve lengths of a vector under the transformation.

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 = 1. \tag{77}$$

The preservation of this relationship within some preassigned tolerance should be assured in any implementation for the e's.

SECTION VIII. CONCLUSION

Four implementations for the calculation of the transformation matrix relating the vehicle-fixed and space-fixed coordinate systems have been outlined. Any of these could be implemented in an advanced guidance system for obtaining navigation information. A study comparing the relative merits of each of these schemes in an actual simulation of an advanced guidance system is the next step.

APPENDIX A

DERIVATION OF THE TRANSFORMATION MATRIX

The transformation matrix to be derived relates two rectangular Cartesian coordinate reference frames with a common origin. The body-fixed reference frame (X_m, Y_m, Z_m) is free to rotate with respect to the inertial reference frame (X_I, Y_I, Z_I) as illustrated in Figure 2.

Now, assume a velocity vector $\vec{V}$ is expressed in terms of its components in the body-fixed coordinate frame. That is

$$\vec{V} = \dot{X}_m \vec{i}_m + \dot{Y}_m \vec{j}_m + \dot{Z}_m \vec{k}_m .$$

This same velocity vector is expressed in terms of its components in the inertial coordinate system as·

$$\vec{V} = \dot{X}_I \vec{i}_I + \dot{Y}_I \vec{j}_I + Z_I \vec{k}_I .$$

Here $\vec{i}_m$, $\vec{j}_m$, $\vec{k}_m$ are unit vectors in the body-fixed coordinate frame while the $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ are unit vectors in the inertial coordinate frame. Since the same velocity vector is represented in both cases, we may equate the two expressions for $\vec{V}$.

$$\dot{X}_I \vec{i}_I + Y_I \vec{j}_I + \dot{Z}_I \vec{k}_I = \dot{X}_m \vec{i}_m + \dot{Y}_m \vec{j}_m + \dot{Z}_m \vec{k}_m .$$

Taking the scalar or dot product of each term in this equation with $\vec{i}_I$, $\vec{j}_I$, and $\vec{k}_I$ in succession yields the following set of equations·

$$\dot{X}_I = \dot{X}_m (\vec{i}_m \cdot \vec{i}_I) + \dot{Y}_m (\vec{j}_m \cdot \vec{i}_I) + \dot{Z}_m (\vec{k}_m \cdot \vec{i}_I)$$

$$\dot{Y}_I = \dot{X}_m (\vec{i}_m \cdot \vec{j}_I) + \dot{Y}_m (\vec{j}_m \cdot \vec{j}_I) + \dot{Z}_m (\vec{k}_m \cdot \vec{j}_I)$$

$$\dot{Z}_I = \dot{X}_m (\vec{i}_m \cdot \vec{k}_I) + \dot{Y}_m (\vec{j}_m \cdot \vec{k}_I) + \dot{Z}_m (\vec{k}_m \cdot \vec{k}_I) .$$

The indicated dot products in parentheses are just the cosines of the angles between the unit vectors. This follows from the definitions of the scalar or dot product and the unit vector.

By taking the dot product of each term of the same equation as before with $\vec{i}_m$, $\vec{j}_m$, and $\vec{k}_m$ in succession, the following set of equations is derived.

$$\dot{X}_m = \dot{X}_I (\vec{i}_I \cdot \vec{i}_m) + \dot{Y}_I (\vec{j}_I \cdot \vec{i}_m) + \dot{Z}_I (\vec{k}_I \cdot \vec{i}_m)$$

$$\dot{Y}_m = \dot{X}_I (\vec{i}_I \cdot \vec{j}_m) + \dot{Y}_I (\vec{j}_I \cdot \vec{j}_m) + \dot{Z}_I (\vec{k}_I \cdot \vec{j}_m)$$

$$\dot{Z}_m = \dot{X}_I (\vec{i}_I \cdot \vec{k}_m) + \dot{Y}_I (\vec{j}_I \cdot \vec{k}_m) + \dot{Z}_I (\vec{k}_I \cdot \vec{k}_m).$$

Again, the dot products indicated are the cosines of the angles between the unit vectors.

If we let c_{11} represent dot product $(\vec{i}_m \cdot \vec{i}_I)$, then it also represents the dot product $(\vec{i}_I \cdot \vec{i}_m)$ since the dot product is commutative. Letting a unique c_{ij} represent each unique dot product, the two sets of transformation equations may be written in matrix form as:

$$\begin{pmatrix} \dot{X}_I \\ \dot{Y}_I \\ \dot{Z}_I \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{X}_m \\ \dot{Y}_m \\ \dot{Z}_m \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{X}_m \\ \dot{Y}_m \\ \dot{Z}_m \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{X}_I \\ \dot{Y}_I \\ Z_I \end{pmatrix}.$$

If we let C represent the transformation matrix in the first matrix equation, we note that C is the matrix we are seeking. Note also that the transformation matrix in the second matrix equation is the transpose of C. In this matrix equation C^t fills the role of C inverse; therefore C must be an orthogonal transformation matrix.

APPENDIX B

PROPERTIES OF ORTHOGONAL TRANSFORMATION MATRICES

The transformation that carries the measured velocities in the vehicle coordinate system into the components of velocity in the inertial coordinate system is a proper rotation. This is true since the two coordinate systems have a common origin and both are right-handed rectangular Cartesian coordinate systems. The inertial coordinate system has a particular space-fixed orientation while the vehicle-fixed coordinate system is free to rotate in any manner about the common origin.

The magnitude of the total velocity vector must be the same in both coordinate systems which we can write symbolically as:

$$\left|\vec{V}_m\right| = \left|\vec{V}_I\right|$$

where $\left|\vec{V}_m\right| = (\dot{X}_m^2 + \dot{Y}_m^2 + \dot{Z}_m^2)^{\frac{1}{2}}$, $\left|\vec{V}_I\right| = (\dot{X}_I^2 + \dot{Y}_I^2 + \dot{Z}_I^2)^{\frac{1}{2}}$.

Therefore: $(\dot{X}_m^2 + \dot{Y}_m^2 + \dot{Z}_m^2)^{\frac{1}{2}} = (\dot{X}_I^2 + \dot{Y}_I^2 + \dot{Z}_I^2)^{\frac{1}{2}}$.

It follows that: $\dot{X}_m^2 + \dot{Y}_m^2 + \dot{Z}_m^2 = \dot{X}_I^2 + \dot{Y}_I^2 + \dot{Z}_I^2$.

The transformation under study may be written as follows:

$$\vec{V}_s = C\vec{V}_I$$

which may be written out as

$$\begin{pmatrix} \dot{X}_I \\ \dot{Y}_I \\ \dot{Z}_I \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{X}_m \\ \dot{Y}_m \\ \dot{Z}_m \end{pmatrix}.$$

See Appendix A for derivation of the C matrix.

Thus, from this matrix equation, solving for $\dot{X}_I$:

$$\dot{X}_I = c_{11}\,\dot{X}_m + c_{12}\,\dot{Y}_m + c_{13}\,\dot{Z}_m .$$

Squaring both sides:

$$\dot{X}_I^2 = c_{11}^2\,\dot{X}_m^2 + c_{12}^2\,\dot{Y}_m^2 + c_{13}\,\dot{Z}_m^2 + 2\,c_{11}\,c_{12}\,\dot{X}_m\,\dot{Y}_m$$
$$+ 2\,c_{11}\,c_{13}\,\dot{X}_m\,\dot{Z}_m + 2\,c_{12}\,c_{13}\,\dot{Y}_m\,\dot{Z}_m .$$

Solving for $\dot{Y}_I$ from the matrix equation:

$$\dot{Y}_I = c_{21}\,\dot{X}_m + c_{22}\,\dot{Y}_m + c_{23}\,\dot{Z}_m .$$

Squaring both sides:

$$\dot{Y}_I^2 = c_{21}^2\,\dot{X}_m^2 + c_{22}^2\,\dot{Y}_m^2 + c_{23}^2\,\dot{Z}_m^2 + 2\,c_{21}\,c_{22}\,\dot{X}_m\,\dot{Y}_m$$
$$+ 2\,c_{21}\,c_{23}\,\dot{X}_m\,\dot{Z}_m + 2\,c_{22}\,c_{23}\,\dot{Y}_m\,\dot{Z}_m .$$

Solving for $\dot{Z}_I$ from the matrix equation:

$$\dot{Z}_I = c_{31}\,\dot{X}_m + c_{32}\,\dot{Y}_m + c_{33}\,\dot{Z}_m .$$

Squaring both sides:

$$\dot{Z}_I^2 = c_{31}^2\,\dot{X}_m^2 + c_{32}^2\,\dot{Y}_m^2 + c_{33}^2\,\dot{Z}_m^2 + 2\,c_{31}\,c_{32}\,\dot{X}_m\,\dot{Y}_m$$
$$+ 2\,c_{31}\,c_{33}\,\dot{X}_m\,\dot{Z}_m + 2\,c_{32}\,c_{33}\,\dot{Y}_m\,\dot{Z}_m .$$

Adding the squared inertial velocity components:

$$\dot{X}_I^2 + \dot{Y}_I^2 + \dot{Z}_I^2 = (c_{11}^2 + c_{21}^2 + c_{31}^2)\ \dot{X}_m^2 + (c_{12}^2 + c_{22}^2 + c_{32}^2)\ \dot{Y}_m^2$$

$$+ (c_{13}^2 + c_{23}^2 + c_{33}^2)\ \dot{Z}_m^2 + 2\,(c_{11}\,c_{12} + c_{21}\,c_{22} + c_{31}\,c_{32})\,\dot{X}_m\,\dot{Y}_m$$

$$+ 2\,(c_{11}\,c_{13} + c_{21}\,c_{23} + c_{31}\,c_{32})\,\dot{X}_m\,\dot{Z}_m$$

$$+ 2\,(c_{12}\,c_{13} + c_{22}\,c_{23} + c_{32}\,c_{33})\,\dot{Y}_m\,\dot{Z}_m .$$

Another expression for the sum of the squared inertial velocity term is:

$$\dot{X}_I^2 + \dot{Y}_I^2 + \dot{Z}_I^2 = \dot{X}_m^2 + \dot{Y}_m^2 + \dot{Z}_m^2 .$$

Equating the coefficients of the velocity term on the right-hand sides of these two equivalent expressions gives the following relations among the elements of C:

$$\left.\begin{aligned} c_{11}^2 + c_{21}^2 + c_{31}^2 &= 1 \\ c_{12}^2 + c_{22}^2 + c_{32}^2 &= 1 \\ c_{13}^2 + c_{23}^2 + c_{33}^2 &= 1 \end{aligned}\right\}$$ 3 normality conditions

and

$$\left.\begin{aligned} c_{11}\,c_{12} + c_{21}\,c_{22} + c_{31}\,c_{32} &= 0 \\ c_{11}\,c_{13} + c_{21}\,c_{23} + c_{31}\,c_{33} &= 0 \\ c_{12}\,c_{13} + c_{22}\,c_{23} + c_{32}\,c_{33} &= 0 \end{aligned}\right\}$$ 3 orthogonality conditions.

Since the transformation is a proper rotation, the matrix C representing the transformation is a proper orthogonal transformation matrix. Two properties of proper orthogonal transformation matrices are·

(a) The inverse of C is equal to the transpose of C.

(b) The determinant of C is equal to 1.

A direct consequence of property (a) is that we may write:

$$\begin{pmatrix} \dot{X}_m \\ \dot{Y}_m \\ \dot{Z}_m \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} \dot{X}_I \\ \dot{Y}_I \\ \dot{Z}_I \end{pmatrix}.$$

See Appendix A for a derivation of this matrix equation.

Thus

$$\dot{X}_m = c_{11}\,\dot{X}_I + c_{21}\,\dot{Y}_I + c_{31}\,\dot{Z}_I$$

and

$$\dot{X}_m^2 = c_{11}^2\,X_I^2 + c_{21}^2\,Y_I^2 + c_{31}^2\,\dot{Z}_I^2 + 2\,c_{11}\,c_{21}\,\dot{X}_I\,\dot{Y}_I$$
$$+ 2\,c_{11}\,c_{31}\,\dot{X}_I\,\dot{Z}_I + 2\,c_{21}\,c_{31}\,\dot{Y}_I\,\dot{Z}_I.$$

Similarly

$$\dot{Y}_m = c_{12}\,\dot{X}_I + c_{22}\,\dot{Y}_I + c_{32}\,\dot{Z}_I$$

$$\dot{Y}_m^2 = c_{12}^2\,\dot{X}_I^2 + c_{22}^2\,\dot{Y}_I^2 + c_{32}^2\,\dot{Z}_I^2 + 2\,c_{12}\,c_{22}\,\dot{X}_I\,\dot{Y}_I$$
$$+ 2\,c_{12}\,c_{32}\,\dot{X}_I\,\dot{Z}_I + 2\,c_{22}\,c_{32}\,\dot{Y}_I\,\dot{Z}_I$$

and

$$\dot{Z}_m = c_{13}\,\dot{X}_I + c_{23}\,\dot{Y}_I + c_{33}\,\dot{Z}_I$$

$$\dot{Z}_m^2 = c_{13}^2\,\dot{X}_I^2 + c_{23}^2\,\dot{Y}_I^2 + c_{33}^2\,\dot{Z}_I^2 + 2\,c_{13}\,c_{23}\,\dot{X}_I\,\dot{Y}_I$$
$$+ 2\,c_{13}\,c_{33}\,\dot{X}_I\,\dot{Z}_I + 2\,c_{23}\,c_{33}\,\dot{Y}_I\,\dot{Z}_I.$$

Adding the squared body-fixed velocity term:

$$\dot{X}_m^2 + \dot{Y}_m^2 + \dot{Z}_m^2 = (c_{11}^2 + c_{12} + c_{13}^2)\ \dot{X}_I^2 + (c_{21}^2 + c_{22}^2 + c_{23}^2)\ \dot{Y}_I^2$$

$$+ (c_{31}^2 + c_{32}^2 + c_{33}^2)\ \dot{Z}_I^2 + 2\,(c_{11}\,c_{21} + c_{12}\,c_{22} + c_{13}\,c_{23})\ \dot{X}_I\,\dot{Y}_I$$

$$+ 2\,(c_{11}\,c_{31} + c_{12}\,c_{32} + c_{13}\,c_{33})\ \dot{X}_I\,\dot{Z}_I$$

$$+ 2\,(c_{21}\,c_{31} + c_{22}\,c_{32} + c_{23}\,c_{33})\ \dot{Y}_I\,\dot{Z}_I$$

and since $\dot{X}_m^2 + \dot{Y}_m^2 + \dot{Z}_m^2 = \dot{X}_I^2 + \dot{Y}_I^2 + \dot{Z}_I^2$

then

$$\left.\begin{aligned} c_{11}^2 + c_{12}^2 + c_{13}^2 &= 1 \\ c_{21}^2 + c_{22}^2 + c_{23}^2 &= 1 \\ c_{31}^2 + c_{32}^2 + c_{33}^2 &= 1 \end{aligned}\right\}$$ 3 normality conditions

and

$$\left.\begin{aligned} c_{11}\,c_{21} + c_{12}\,c_{22} + c_{13}\,c_{23} &= 0 \\ c_{11}\,c_{31} + c_{12}\,c_{32} + c_{13}\,c_{33} &= 0 \\ c_{21}\,c_{31} + c_{22}\,c_{32} + c_{23}\,c_{33} &= 0 \end{aligned}\right\}$$ 3 orthogonality conditions.

These 6 equations and the 6 equations derived for the inverse transformation give 12 conditions for the C matrix. These orthogonality and normality conditions hold for any orthogonal transformation matrix.

Ten more relations between the elements of proper orthogonal transformation matrices may be found. Since the determinant is equal to 1, the following relation holds·

$$c_{11}\,c_{22}\,c_{33} - c_{11}\,c_{23}\,c_{32} + c_{12}\,c_{23}\,c_{31} - c_{12}\,c_{21}\,c_{33} + c_{13}\,c_{21}\,c_{32}$$

$$- c_{13}\,c_{22}\,c_{31} = 1.$$

As a result of the property that the transpose of C is equal to the inverse of C, we may equate each individual element of C^t to each individual element of C^{-1}. This will give us nine additional relations between the elements of C.

First, the transpose of C is formed.

$$C^t = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} .$$

Then, we begin to derive the inverse of C by replacing each element of C by its cofactor:

$$\begin{pmatrix} c_{22}c_{33} - c_{23}c_{32} & c_{23}c_{31} - c_{21}c_{33} & c_{21}c_{32} - c_{22}c_{31} \\ c_{13}c_{32} - c_{12}c_{33} & c_{11}c_{33} - c_{31}c_{13} & c_{12}c_{31} - c_{11}c_{32} \\ c_{12}c_{23} - c_{13}c_{22} & c_{13}c_{21} - c_{11}c_{23} & c_{11}c_{22} - c_{12}c_{21} \end{pmatrix} .$$

Taking the transpose of this matrix we arrive at the inverse.

$$C^{-1} = \begin{pmatrix} c_{22}c_{33} - c_{23}c_{32} & c_{13}c_{32} - c_{12}c_{33} & c_{12}c_{23} - c_{13}c_{22} \\ c_{23}c_{31} - c_{21}c_{33} & c_{11}c_{33} - c_{13}c_{31} & c_{13}c_{21} - c_{11}c_{23} \\ c_{21}c_{32} - c_{22}c_{31} & c_{12}c_{31} - c_{11}c_{32} & c_{11}c_{22} - c_{12}c_{21} \end{pmatrix} .$$

Equating corresponding elements of C^t and C^{-1} gives the following nine relations which may be considered as three sets:

$$c_{11} = c_{22}\, c_{33} - c_{23}\, c_{32}$$

$$c_{12} = c_{23}\, c_{31} - c_{21}\, c_{33}$$

$$c_{13} = c_{21}\, c_{32} - c_{22}\, c_{31}$$

$$c_{21} = c_{13}\, c_{32} - c_{12}\, c_{33}$$

$$c_{22} = c_{11}\, c_{33} - c_{13}\, c_{31}$$

$$c_{23} = c_{12}\, c_{31} - c_{11}\, c_{32}$$

$$c_{31} = c_{12}\, c_{23} - c_{13}\, c_{22}$$

$$c_{32} = c_{13}\, c_{21} - c_{11}\, c_{23}$$

$$c_{33} = c_{11}\, c_{22} - c_{12}\, c_{21}.$$

From these nine relations it is easy to show that given any two rows or columns of C it is possible to find the third row or column. This is an example of the high degree of interdependence between the C matrix elements.

APPENDIX C

PROOF OF EULER'S THEOREM

Consider a proper orthogonal transformation matrix A. Then, another way of expressing Euler's theorem is as follows: for every A there exists some vector $\vec{R}$ whose components are unchanged by the transformation A. There must be some $\vec{R}$ such that

$$A\vec{R} = \vec{R} \tag{1}$$

for every proper orthogonal transformation matrix A.

Let $\vec{R} = (X, Y, Z)$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Then equation 1 may be written:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} . \tag{2}$$

Multiplying on the right-hand side we obtain the following set of equations:

$$\begin{aligned} a_{11} X + a_{12} Y + a_{13} Z &= X \\ a_{21} X + a_{22} Y + a_{23} Z &= Y \\ a_{31} X + a_{32} Y + a_{33} Z &= Z \ , \end{aligned} \tag{3}$$

which simplifies to the following set of linear homogeneous equations:

$$\begin{aligned} (a_{11} - 1) X + a_{12} Y + a_{13} Z &= 0 \\ a_{21} X + (a_{22} - 1) Y + a_{23} Z &= 0 \\ a_{31} X + a_{32} Y + (a_{33} - 1) Z &= 0 \ . \end{aligned} \tag{4}$$

Recall that a system of n linear homogeneous equations in n unknowns has a nontrivial solution if and only if the determinant of the coefficients is zero. Therefore, if the determinant of the coefficients of equation 4 is zero, Euler's theorem is proven. In other words, we must show that:

$$\begin{vmatrix} a_{11} - 1 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{vmatrix} = 0. \tag{5}$$

The left-hand determinant of equation 5 may be written as the difference of two determinants:

$$\begin{vmatrix} a_{11} - 1 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{6}$$

$$\begin{vmatrix} a_{11} - 1 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{vmatrix} = |A| - |I|. \tag{7}$$

Note that the left-hand determinant of the right-hand side of equation 6 is $|A|$, and the right-hand determinant of the right-hand side of equation 6 is the determinant of the identity matrix. Since the determinant of a proper orthogonal transformation matrix must equal one and the determinant of the identity matrix is also one, equation 7 will read with these substitutions:

$$\begin{vmatrix} a_{11} - 1 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{vmatrix} = 1 - 1 = 0 \tag{8}$$

and thus Euler's theorem is proven.

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APPROVAL

METHODS OF COMPUTING THE TRANSFORMATION MATRIX ASSOCIATED WITH GIMBALLESS INERTIAL MEASUREMENT UNITS

By Dexter H. Burdeshaw

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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